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General Analysis of Phases in Quark Mass Matrices

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Abstract

We give a detailed discussion of our general determination of (i) the number of unremovable, physically meaningful phases in quark mass matrices and (ii) which elements of these matrices can be rendered real by rephasings of fermion fields. The results are applied to several currently viable models. New results are presented for an arbitrary number of fermion generations; these provide further insight into the three-generation case of physical interest.

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1 Introduction

Understanding fermion masses and quark mixing remains one of the most important outstanding problems in particle physics. In an effort to gain insight into this problem, many studies of simple models of quark mass matrices have been carried out over the years. The phases in these mass matrices play an essential role in the Kobayashi-Maskawa (KM) mechanism [1] for CP violation[1]. A given model is characterized by the number of parameters (amplitudes and phases) which specify the quark mass matrices. Thus, a very important problem is to determine, for any model, how many unremovable, and hence physically meaningful, phases occur in the quark mass matrices and which elements of these matrices can be made real by rephasings of quark fields. We recently presented a general solution to this problem [2]. Here we give a detailed discussion of our results and proofs, and applications to currently viable models. To make the paper self-contained, we will review the results in [2]. In passing, it should be mentioned that we have also given a general solution to the analogous problem for lepton mass matrices[3][2]. The organization of this paper is as follows. In section 2 we discuss our theorem on the number of unremovable phases. In section 3 we give the details of our complex rephasing invariants and results on which elements of the mass matrices can be rendered real. This section contains the proofs of certain theorems presented in Ref. [2]. Section 4 contains some new results on complex invariants and unremovable phases for arbitrary N_G. In section 5 we apply our general results to models.

[1]As will be discussed further below, there are, in general, more unremovable phases in the quark mass matrices than in the quark mixing matrix, so that some of the former phases contribute to CP-conserving quantities. We also note that our analysis does not assume that the KM mechanism is the only source of CP violation.

[2]The situation in the leptonic sector is qualitatively more complicated than that in the quark sector because of the general presence of three types of fermion bilinears, Dirac, left-handed Majorana, and right-handed Majorana, each with its own gauge and rephasing properties. It is also different because one does not know the full set of fields which might be present: there might or might not exist electroweak singlet neutrinos. The most general case was treated in Ref. [3]. The uncertainty in the field content of the neutrino sector produces the same uncertainty in a general analysis of phases for both the quark and lepton sectors considered together. An analysis of invariants and their constraints in a particular unified model was also given in Ref. [3].

2 Theorem on the Number of Physical Phases

The quark mass terms are taken to arise from interactions which are invariant under the standard model gauge group $G_{SM} = SU(3) \times SU(2) \times U(1)$, via the spontaneous symmetry breaking of G_{SM}. In the standard model and its supersymmetric extensions, the resultant mass terms appear at the electroweak level via (renormalizable, dimension-4) Yukawa couplings[3,4]. It follows that these mass terms can be written in terms of the G_{SM} quark fields as

$$-\mathcal{L}_m = \sum_{j,k=1}^{N_G} \Big[(\bar{Q}_{jL})_1 M^{(u)}_{jk} u_{kR} + (\bar{Q}_{jL})_2 M^{(d)}_{jk} d_{kR} \Big] + h.c. \tag{2.1}$$

where j and k are generation labels; $N_G = 3$ is the number of generations of standard-model fermions; Q_{jL} is an SU(2) doublet, with $Q_{1L} = {u \choose d}_L$, $Q_{2L} = {c \choose s}_L$, $Q_{3L} = {t \choose b}_L$; the subscript a on $(Q_{jL})_a$ is the SU(2) index; and the SU(2)-singlet right-handed quark fields are denoted as u_{kR} with $u_{1R} = u_R$, $u_{2R} = c_R$, $u_{3R} = t_R$, $d_{1R} = d_R$, $d_{2R} = s_R$, $d_{3R} = b_R$. As indicated, we will concentrate on the physical case of $N_G = 3$ generations of standard-model fermions (with associated light neutrinos). However, for the sake of generality, and because it provides further insight into the physical $N_G = 3$ case, we will give a number of results for arbitrary N_G. $M^{(u)}$ and $M^{(d)}$ are the mass matrices in the up and down sectors, whose diagonalization yields the mass eigenstates u_{jm} and d_{jm}. Because of the assumed origin of the quark mass

[3] We do not consider models in which fermion masses arise via multifermion operators, which are not perturbatively renormalizable. Note that at a nonperturbative level, lattice studies [4] show that a lattice theory with a specific multifermion action and no scalar fields may yield the same continuum limit as a theory with a Yukawa interaction.

[4] In a number of interesting models, these Yukawa couplings are viewed as originating at a higher mass scale, such as that of a hypothetical supersymmetric grand unified theory (SUSY GUT) or some other (supersymmetric) theory resulting from the $E << (\alpha')^{-1/2}$ limit of a string theory. It is possible that interactions which appear to be Yukawa couplings at a given mass scale, could be effective, in the sense that some of the elements of the associated Yukawa matrices could actually arise from higher-dimension operators at a yet higher mass scale. An example of a higher-dimension operator which might plausibly occur near the scale of quantum gravity is a dimension-5 operator of the generic form $(c/\bar{M}_P)\phi_1\phi_2\bar{Q}_L q_R$, where ϕ_1 and ϕ_2 are appropriate Higgs fields, c denotes a dimensionless coefficient, and $\bar{M}_P$ is the (reduced) Planck mass, $\bar{M}_P = \sqrt{\hbar c/(8\pi G_N)}$. The plausibility of such higher-dimension operators can be inferred either from the nonrenormalizability of supergravity or as a consequence of the $E << (\alpha')^{-1/2}$ limit of a string theory. At lower mass scales this operator could produce effective Yukawa terms contributing to (2.1), where, say, ϕ_1 contains H_i, $i = 1$ or 2 as defined in (2.3), (2.4), and the factor $c < \phi_2 > /\bar{M}_P$ enters into the effective Yukawa or mass matrix (where $< \phi_2 >$ might have a value somewhat smaller than $\bar{M}_P$ but much larger than the electroweak scale).

matrices from Yukawa interactions (i.e. interactions which appear as dimension-4 at a given mass scale, but which could include the contributions of higher-dimension operators at higher mass scales), it is convenient to write these mass matrices in terms of dimensionless Yukawa matrices according to

$$Y^{(f)} = 2^{1/2} v_f^{-1} M^{(f)} \ , \quad f = u, d \tag{2.2}$$

where v_u and v_d are real quantities with the dimensions of mass, and represent Higgs vacuum expectation values in the standard model and its supersymmetic extensions. For example, in the minimal supersymmetric standard model (MSSM),

$$< H_1 >= \begin{pmatrix} 0 \\ 2^{-1/2} v_d \end{pmatrix} \tag{2.3}$$

and

$$< H_2 >= \begin{pmatrix} 2^{-1/2} v_u \\ 0 \end{pmatrix} \tag{2.4}$$

where H_1 and H_2 are the $Y = 1$ and $Y = -1$ Higgs fields. With one definition for the angle β in the MSSM, $v_u = v \sin\beta$, $v_d = v \cos\beta$, where $v \equiv 2^{-1/4} G_F^{-1/2} = 246$ GeV (many authors use a different convention according to which $v_u = v \cos\beta$, $v_d = v \sin\beta$). Because v_u and v_d are real, the phase properties of the mass matrices $M^{(f)}$ are the same as those of the Yukawa matrices $Y^{(f)}$ and we shall deal with them interchangeably.[5]

To count the number of unremovable, and hence physically meaningful, phases in the quark mass matrices, we rephase the fermion fields in (2.1) so as to remove all possible phases in these matrices.

A general theorem on this counting problem was given before [2]; we review it here. As was evident in Ref. [2], this theorem applies for arbitrary N_G; although the physically interesting case is $N_G = 3$, we indicate the general N_G-dependence here. One can perform the rephasings defined by

$$Q_{jL} = e^{-i\alpha_j} Q'_{jL} \tag{2.5}$$

[5]In specific models in which the Yukawa couplings arise at a high mass scale, e.g. in a hypothetical grand unified theory or an effective pointlike field theory arising from the $E << (\alpha')^{-1/2}$ limit of a string theory, and in which below this scale the gauge group reduces to G_{SM} without further intermediate structure, it is the $Y^{(f)}$ which one evolves down to the electroweak scale using the appropriate renormalization group equations. Note that $M^{(f)}$ and $Y^{(f)}$ as defined above may subsume the effects of several different types of Yukawa couplings of fermion and Higgs representations in the theory at the high mass scale, e.g. a GUT.

$$u_{jR} = e^{i\beta_j^{(u)}} u'_{jR} \tag{2.6}$$

$$d_{jR} = e^{i\beta_j^{(d)}} d'_{jR} \tag{2.7}$$

for $1 \leq j \leq N_G$. (The minus sign in (2.5) is included for technical convenience to avoid minus signs in later formulas.) In passing, we note that since the mass terms in eq. (2.1) are assumed to have arisen from G_{SM}-invariant Yukawa interactions, the Yukawa matrices, as coefficients of G_{SM}-invariant operators, are obviously left invariant by G_{SM} gauge transformations. Also, note that since at the level of (2.1) which defines the quark mass matrices, no explicit Higgs fields appear, we do not apply the rephasings of these Higgs fields. In any case, in particular models, such as the MSSM, the original Higgs doublets H_1 and H_2 have already been rephased so as to yield real vacuum expectation values v_d and v_u. In terms of the primed (rephased) fermion fields, the mass matrices, or equivalently the Yukawa matrices, have elements

$$Y_{jk}^{(f)\prime} = e^{i(\alpha_j + \beta_k^{(f)})} Y_{jk}^{(f)} \tag{2.8}$$

for $f = u, d$. Thus, if $Y^{(f)}$ has N_f nonzero, and, in general, complex elements, then the N_f equations for making these elements real are

$$\alpha_j + \beta_k^{(f)} = -\arg(Y_{jk}^{(f)}) + \eta_{jk}^{(f)} \pi \tag{2.9}$$

for $f = u, d$, where the set $\{jk\}$ runs over each of these nonzero elements, and $\eta_{jk}^{(f)} = 0$ or 1. The η_{jk} term allows for the possibility of making the rephased element real and negative rather than positive; this will not affect the counting of unremovable phases. Let us define the vector of fermion field phases

$$v = (\{\alpha_i\}, \{\beta_i^{(u)}\}, \{\beta_i^{(d)}\})^T \tag{2.10}$$

of dimension $3N_G$, where $\{\alpha_i\} \equiv \{\alpha_1, ..., \alpha_{N_G}\}$, $\{\beta_i^{(f)}\} \equiv \{\beta_1^{(f)}, ..., \beta_{N_G}^{(f)}\}$ for $f = u, d$, and

$$w = (\{-\arg(Y_{jk}^{(u)}) + \eta_{jk}^{(u)} \pi\}, \ \{-\arg(Y_{mn}^{(d)}) + \eta_{mn}^{(d)} \pi\})^T \tag{2.11}$$

of dimension equal to the number of rephasing equations $N_{eq} = N_u + N_d$, where the in-

dices jk and mn take appropriate values.[6] For explicit calculations, the ordering of the elements of the vector w is given by the sequence of nonzero elements $Y^{(u)}_{jk}$ for $jk = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ followed by the nonzero elements $Y^{(d)}_{mn}$ with $mn =$ ranging over the same set, $\{11, 12, 13, 21, 22, 23, 31, 32, 33\}$. We can then write (2.9) for $f = u, d$ as

$$Tv = w \tag{2.13}$$

which defines the N_{eq}-row by $3N_G$-column matrix T. Since a nonzero element of $Y^{(f)}$, $f = u, d$ is, in general complex (because no symmetry requires it to be real), there is an obvious general relation between the number of rephasing equations N_{eq}, its maximal value, $(N_{eq})_{max} = 2N_G^2$, and the number of zero elements, N_z, in the Yukawa matrices:

$$N_{eq} = 2N_G^2 - N_z \tag{2.14}$$

In studying specific models, it is also helpful to introduce the notation $N_{z,f}$ for the number of zeroes in each of the Yukawa matrices $Y^{(f)}$, for $f = u$ and $f = d$. Obviously, $N_z = N_{z,u} + N_{z,d}$. If an element of $Y^{(f)}$, $f = u, d$ is zero at some energy scale E, this will not necessarily be true at a different energy scale E', so that in specifying the numbers $N_{z,u}$ and $N_{z,d}$, as well as in specifying the forms of the Yukawa matrices $Y^{(u)}$ and $Y^{(d)}$ themselves, one must indicate the energy scale involved. If at this energy scale, the gauge group and the structure of the Yukawa couplings guarantee that the $Y^{(f)}$, $f = u, d$, are (complex) symmetric, then, of course, $Y^{(f)}_{jk} = 0$ is equivalent to $Y^{(f)}_{kj} = 0$; however, it will be convenient to count both in our definition of $N_{z,f}$ and N_z.

Since T is an N_{eq}-row by $3N_G$-column matrix, clearly

$$rank(T) < \min\{N_{eq},\ 3N_G\} \tag{2.15}$$

[6]In eq. (2.11) we change notation from Ref. [2], where an overall minus sign was absorbed in T, so that the analogue of eq. (2.11) was

$$w = (\{\arg(Y^{(u)}) + \eta^{(u)}_{jk}\pi\},\ \{\arg(Y^{(d)}) + \eta^{(d)}_{jk}\pi\})^T \tag{2.12}$$

(recall that since the arguments are defined mod 2π and $\eta^{(f)}_{jk} = 0$ or 1, it follows that $\eta^{(f)}_{jk}\pi = -\eta^{(f)}_{jk}\pi$ (mod 2π).) The present definition avoids minus signs in our explicit listings of T matrices below (e.g. eq. (5.1.3) etc.).

For the case (realized in all experimentally viable models which we have studied) where $N_{eq} \geq 3N_G$, one has a more restrictive upper bound on $rank(T)$:

$$rank(T) \leq 3N_G - 1, \qquad (for \ \ N_{eq} \geq 3N_G) \tag{2.16}$$

This is proved by ruling out the only other possibility, i.e. $rank(T) = 3N_G$. The reason that $rank(T)$ cannot have its apparently maximal value is that one overall rephasing has no effect on the Yukawa interaction, namely the U(1) generated by (2.5)-(2.7) with $-\alpha_i = \beta_j^{(u)} = \beta_k^{(d)}$ for all i, j, k. For the relevant case $N_G = 3$, for all of the (realistic) models that we have studied, the upper bound (2.16) is saturated, i.e. $rank(T) = 8$. This is not a general result; we will exhibit a toy model (model 7 in section 5) where $rank(T) = 7$, although that model is a degenerate case.

The main theorem [2] is as follows:

Theorem:

The number of unremovable phases N_p in $Y^{(u)}$ and $Y^{(d)}$ is

$$N_p = N_{eq} - rank(T) \tag{2.17}$$

(which holds for arbitrary N_G).

Proof:

First, denote $rank(T) = r_{_T}$. Then one can select and delete $N_{eq} - r_{_T}$ rows from the matrix T without reducing the rank of the resultant matrix. This deletion means that we do not attempt to remove the phases from the corresponding elements of the $Y^{(f)}$, $f = u, d$. We perform this reduction. For the remaining $r_{_T}$ equations, we move a subset of $3N_G - r_{_T}$ phases in v to the right-hand side of (2.9), thus including them in a redefined $\bar{w}$. This yields a set of $r_{_T}$ linear equations in $r_{_T}$ unknown phases, denoted $\bar{v}$. We write this as $\bar{T}\bar{v} = \bar{w}$. Since by construction $rank(\bar{T}) = r_{_T}$, it follows that $\bar{T}$ is invertible, so that one can now solve for the $r_{_T}$ fermion rephasings in $\bar{v}$ which render $r_{_T}$ of the N_{eq} complex elements real: $\bar{v} = \bar{T}^{-1}\bar{w}$. Hence there are $N_{eq} - r_{_T}$ remaining phases in the $Y^{(f)}$, as claimed. $\Box$.

The operations involved in this proof and the construction of $\bar{T}$ will be explicitly illustrated for model 1 in section 5.

Some comments are in order. First, we note that in order for these phases to be physically meaningful, it is, of course, necessary that they be unchanged under the full set of rephasings of fermion fields. In fact, as we will show, there is a one-to-one correspondence between each such unremovable phase and an independent phase of a certain product of elements of mass matrices which is invariant under fermion field rephasings. (In special cases a model may have an unremovable, invariant phase which, because of a particular symmetry, is zero or π.) Second, as is clear from our proof, in general, the result (2.17) does not depend on whether or not $Y^{(f)}_{jk} = Y^{(f)}_{kj}$ initially. Hence making $Y^{(f)}$ (complex) symmetric for either $f = u$ or $f = d$ does not, in general, result in any reduction in N_p, since in terms of the G_{SM} theory, nothing guarantees this symmetry, and the rephasing is carried out in terms of G_{SM} fields. Third, if one of the unremovable phases is put in a given off-diagonal $Y^{(f)}_{jk}$, one may wish to modify the $kj'th$ equation to read

$$\alpha_k + \beta^{(f)}_j = -\arg(Y^{(f)}_{kj}) - \arg(Y^{(f)}_{jk}) \tag{2.18}$$

For example, in a model where $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$, this would yield $Y^{(f)*}_{jk} = Y^{(f)}_{kj}$ for this pair jk. The modification in (2.18) has no effect on the counting of phases.

Using our theorem, we may calculate the maximal number of unremovable phases N_p in the Yukawa matrices as a function of the number of fermion generations, N_G. For this purpose, we consider the case in which these matrices have all nonzero arbitrary (and, hence in general, complex) elements. Then N_{eq} takes on its maximal value,

$$(N_{eq})_{max} = 2N_G^2 \tag{2.19}$$

Since the elements of the $Y^{(f)}$ are arbitrary, it follows that they satisfy no special relations, so that the T matrix has maximal rank. By our result (2.16), this is $rank(T)_{max} = 3N_G - 1$. Using our theorem (2.17), we thus find that the maximal number of unremovable phases is

$$(N_p)_{max} = (N_G - 1)(2N_G - 1) \tag{2.20}$$

That this case yields the maximal number of unremovable phases is clear, since we can obtain any other form by making some entries of the Yukawa matrices zero, and this can

only decrease the number of unremovable phases. For $N_G = 1, 2$ and 3, $(N_p)_{max}$ takes the values 0, 3, and 10, respectively. In current phenomenological models of quark mass matrices, one tries to make as many entries in the Yukawa matrices as possible zero to minimize the number of parameters and increase predictiveness, so that in the physical $N_G = 3$ case, we will find that N_p is typically just 2 or 3.

3 Rephasing Invariants and Theorems on Locations of Phases

A fundamental question concerns which elements of $Y^{(u)}$ and $Y^{(d)}$ can be made real by fermion rephasings. This is connected with the issue of which rows are to be removed from T to obtain $\bar{T}$, i.e. which nonzero elements of the $Y^{(u)}$ and $Y^{(d)}$ are left complex. In Ref. [2] several theorems were given which provide a general answer to these questions. We review these theorems here and give the details of the proofs. As in the previous section, for generality, we shall indicate which results apply for an arbitrary number of fermion generations, N_G. In the next section we shall also give a number of new results for arbitrary N_G. The general method is to construct all independent complex products of elements of the $Y^{(f)}$, $f = u, d$, having the property that these products are invariant under the rephasings (2.5)-(2.7). Here we use the term "invariant products" to mean invariant under the rephasings (2.5)-(2.7). These products must involve an even number of such elements, since for each index on a Y, there must be a corresponding index on an Y^* in order to form a rephasing-invariant quantity. Since in general, by construction, these invariant products are complex, i.e. have arguments $\neq 0, \pi$, each one implies a constraint which is that the set of $2n$ elements which comprise it cannot be made simultaneously real by the rephasings (2.5)-(2.7) of the quark fields. Here, by "complex invariant products", we mean products which are, in general, complex for arbitrary $Y^{(f)}$; in special cases, a symmetry of a given model may render some of these complex invariants real. It is easily seen that the only rephasing-invariant products of two elements of the $Y^{(f)}$ must be of the form $|Y^{(f)}_{jk}|^2$, i.e., there are no complex invariant products of order 2. We define an irreducible complex invariant to be one which

cannot be factorized purely into products of lower-order complex invariants. Given that there are no quadratic complex invariants, it follows that all complex invariants of order 4 and 6 are irreducible. As we will prove below, for the physical case of $N_G = 3$ generations of fermions, the phase constraints are totally determined by the complex invariants of order 4 and 6, so in the analysis of phenomenological models, one automatically deals only with irreducible complex invariants.[7] We define a set of independent (irreducible) complex invariants to be a set of complex (irreducible) invariants with the property that no invariant in the set is equal to (i) the complex conjugate of another invariant in the set or (ii) another element in the set with its indices permuted. This does not imply that the arguments of a set of independent (irreducible) complex invariants are linearly independent. Indeed, we will show examples below of sets of independent complex invariants of the same order whose arguments are linearly dependent, and also examples of higher-order complex invariants whose arguments can be expressed as linear combinations of arguments of lower-order complex invariants. We define N_{ia} to denote the number of linearly independent arguments ($\to ia$) among the N_{inv} independent complex invariants. Further, we define N_{inv} to be the total number of independent (irreducible) complex invariants in a given model. It is useful to define $N_{inv,2n}$ to denote the number of independent complex invariants of order $2n$.

We thus construct a set of rephasing-invariant products depending on the up and down quark sectors individually:

$$P^{(f)}_{2n;j_1k_1,\ldots j_nk_n;\sigma_L} = \prod_{a=1}^{n} Y^{(f)}_{j_ak_a} Y^{(f)*}_{\sigma_L(j_a)k_a} \tag{3.1}$$

where $f = u, d$, and σ_L is an element of the permutation group S_n. (This set of invariants is clearly the same as the sets defined by $\prod_{a=1}^{n} Y^{(f)}_{j_ak_a} Y^{(f)*}_{j_a\sigma_R(k_a)}$ and $\prod_{a=1}^{n} Y^{(f)}_{j_ak_a} Y^{(f)*}_{\sigma_L(j_a)\sigma_R(k_a)}$, where $\sigma_R \in S_n$; with no loss of generality, we write it in the form of eq. (3.1).) The first theorem is: The products in (3.1) are invariant under the rephasings of fermion fields (2.5)-(2.7). To

[7] Reducible complex invariants only occur in the hypothetical case $N_G \geq 4$ and then first occur at order 8, as products of different complex quartic invariants. Since reducible invariants do not yield any new phase constraints, it suffices to consider only irreducible complex invariants. Hence in all of our discussion below, in particular, the section on arbitrary N_G, we shall take "complex invariants" to mean "irreducible complex invariants".

prove this, we observe that from eq. (2.8), it follows that under these rephasings,

$$\begin{aligned}\prod_{a=1}^{n} Y^{(f)}_{j_a k_a} Y^{(f)*}_{\sigma_L(j_a)k_a} &\rightarrow \prod_{a=1}^{n} [e^{i(\alpha_{j_a}+\beta^{(f)}_{k_a})} Y^{(f)}_{j_a k_a}][e^{-i(\alpha_{\sigma_L(j_a)}+\beta^{(f)}_{k_a})} Y^{(f)*}_{\sigma_L(j_a)k_a}] \\ &= \left(e^{i\sum_{b=1}^{n}(\alpha_{j_b}-\alpha_{\sigma_L(j_b)})}\right) \prod_{a=1}^{n} Y^{(f)}_{j_a k_a} Y^{(f)*}_{\sigma_L(j_a)k_a} \\ &= \prod_{a=1}^{n} Y^{(f)}_{j_a k_a} Y^{(f)*}_{\sigma_L(j_a)k_a} \end{aligned} \tag{3.2}$$

where $\sum_{b=1}^{n}(\alpha_{j_b}-\alpha_{\sigma_L(j_b)})=0$ because the permutations of S_n are an automorphism of the set $(1,2,...,n)$. Hence, the products $P^{(f)}_{2n;j_1k_1,...j_nk_n;\sigma_L}$ are invariant under rephasings of the quark fields, as claimed. $\Box$.

Secondly, we construct a set of invariants connecting the up and down quark sectors:

$$Q^{(s,t)}_{2n;\{j\},\{k\},\{m\};\sigma_L,\sigma_u;\sigma_d} = (\prod_{a=1}^{s} Y^{(u)}_{j_a k_a})(\prod_{b=1}^{t} Y^{(d)}_{j_{s+b}\ m_b})(\prod_{c=1}^{s} Y^{(u)*}_{\sigma_L(j_c)\sigma_u(k_c)})(\prod_{e=1}^{t} Y^{(d)*}_{\sigma_L(j_{s+e})\sigma_d(m_e)}) \tag{3.3}$$

where $s,t\geq 1$, $s+t=n$, $\sigma_L\in S_n$, $\sigma_u\in S_s$, and $\sigma_t\in S_t$. The proof that these are invariant proceeds in a manner similar to that just given above; hence we do not give it explicitly. At quartic order, $2n=4$, $\sigma_L\in S_2$ in eq. (3.1). We use the standard notation $\binom{12}{12}\equiv 1$ and $\binom{12}{21}\equiv(12)\equiv\tau$ (transposition) for the elements of S_2. For $\sigma_L=\tau$, we obtain the complex invariants

$$\begin{aligned} P^{(f)}_{j_1k_1,j_2k_2} &\equiv P^{(f)}_{4;j_1k_1,j_2k_2;\sigma_L=\tau} \\ &= Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f)*}_{j_2k_1} Y^{(f)*}_{j_1k_2} \end{aligned} \tag{3.4}$$

for $f=u,d$. (For the other choice, $\sigma_L=1$, the resultant invariant product, $P^{(f)}_{4;j_1k_1,j_2k_2;\sigma_L=1} = |Y^{(f)}_{j_1k_1}|^2|Y^{(f)}_{j_2k_2}|^2$ is real and hence yields no constraint on phases.) At this quartic order, there is only one Q-type complex invariant; this has $s=t=1$, $\sigma_L=\tau$, so that in the general notation of (3.3), it is $Q^{(1,1)}_{4;\{j_1,j_2\},\{k_1\},\{m_1\};\sigma_L=\tau,\sigma_u=1,\sigma_d=1}$. For brevity of notation, we denote it simply as

$$\begin{aligned} Q_{j_1k_1,j_2m_1} &\equiv Q^{(1,1)}_{4;\{j_1,j_2\},\{k_1\},\{m_1\};\sigma_L=\tau,\sigma_u=1} \\ &= Y^{(u)}_{j_1k_1} Y^{(d)}_{j_2m_1} Y^{(u)*}_{j_2k_1} Y^{(d)*}_{j_1m_1} \end{aligned} \tag{3.5}$$

(As with the quartic P invariant, the other choice for the permutation, $\sigma_L = 1$, yields the product $|Y^{(u)}_{j_1k_1} Y^{(d)}_{j_2m_1}|^2$, which is real.) Note the index symmetry

$$P^{(f)}_{j_1k_1,j_2k_2} = P^{(f)}_{j_2k_2,j_1k_1} \tag{3.6}$$

and the complex conjugation relations

$$P^{(f)}_{j_1k_1j_2k_2} = P^{(f)*}_{j_1k_2,j_2k_1} \tag{3.7}$$

and

$$Q_{j_1k_1,j_2m_1} = Q^*_{j_2k_1,j_1m_1} \tag{3.8}$$

At order $2n = 6$, we find one independent P-type complex invariant for each quark sector $f = u, d$, and two independent Q-type complex invariants; these were given in Ref. [2]. Here we present the details of the analysis. For a given f, we construct the invariants corresponding to each of the six permutations $\sigma_L \in S_3$. In standard notation, these permutations are listed as

$$S_3 = \{\tbinom{123}{123} \equiv 1,\ \tbinom{123}{213} \equiv (12),\ \tbinom{123}{132} \equiv (23),\ \tbinom{123}{321} \equiv (13),\ \tbinom{123}{231} \equiv (231),\ \tbinom{123}{312} \equiv (312)\} \tag{3.9}$$

Clearly for $\sigma_L = 1$, $P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=1} = |Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f)}_{j_3k_3}|^2$, which is real and hence implies no constraint on phases. For the three transpositions $\sigma_L = (12), (23), (13)$, the resultant product reduces to a real factor times a quartic P invariant; for example, $P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=(12)} = |Y^{(f)}_{j_3k_3}|^2 P^{(f)}_{4;j_1k_1,j_2k_2}$, and so forth for the others. The last two elements of S_3 yield genuine 6'th order complex invariants:

$$P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=(231)} = Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f)}_{j_3k_3} Y^{(f)*}_{j_2k_1} Y^{(f)*}_{j_3k_2} Y^{(f)*}_{j_1k_3} \tag{3.10}$$

and

$$P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=(312)} = Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f)}_{j_3k_3} Y^{(f)*}_{j_3k_1} Y^{(f)*}_{j_1k_2} Y^{(f)*}_{j_2k_3} \tag{3.11}$$

However, only one of these is independent, since they are related according to

$$P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=(231)} = P^{(f)}_{6;j_3k_1,j_1k_2,j_2k_3;\sigma_L=(312)} \tag{3.12}$$

This explicit calculation proves our assertion above, that there is one (independent) 6'th order P-type complex invariant. We may take this to be the one in eq. (3.11), and we denote it in a simple notation as

$$P^{(f)}_{j_1k_1,j_2k_2,j_3k_3} \equiv P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3;\sigma_L=(312)} \tag{3.13}$$

To find the 6'th order complex Q-type invariants, we consider the case $(s,t) = (2,1)$ in eq. (3.3); the other case $(s,t) = (1,2)$ can then be obtained easily by the replacement $Y^{(u)} \leftrightarrow Y^{(d)}$ in the resultant products. Since $\sigma_L \in S_3$, $\sigma_u \in S_2$, and $\sigma_d = 1$, it follows that there are $3! \times 2! = 12$ invariants of this type. Each of these can be labelled by $\sigma_L \otimes \sigma_u$. Clearly the $1 \otimes 1$ and $(12) \otimes (12)$ products are real. Further, we find that the products $1 \otimes (12)$, and $(12) \otimes (12)$ give real factors times quartic P-type invariants, while $(23) \otimes 1$, $(13) \otimes 1$, $(231) \otimes (12)$, and $(312) \otimes (12)$ give real factors times quartic Q-type invariants. The remaining four choices for the permutations give genuine complex 6'th order invariants. To discuss these, we shall introduce a convenient notation,

$$Q^{(2,1)}_{6;j_1k_1,j_2k_2,j_3m_1;\sigma_L,\sigma_u} \equiv Q^{(2,1)}_{6;\{j_1,j_2,j_3\},\{k_1,k_2\},\{m_1\};\sigma_L,\sigma_u(\sigma_d=1)} \tag{3.14}$$

The permutations $(23) \otimes (12)$ and $(312) \otimes 1$ yield the same product,

$$Q^{(2,1)}_{6;j_1k_1,j_2k_2,j_3m_1;\sigma_L=(23),\sigma_u=(12)} = Y^{(u)}_{j_1k_1} Y^{(u)}_{j_2k_2} Y^{(d)}_{j_3m_1} Y^{(u)*}_{j_1k_2} Y^{(u)*}_{j_3k_1} Y^{(d)*}_{j_2m_1} \tag{3.15}$$

while the choices $(13) \otimes (12)$ and $(231) \otimes 1$ yield the same product,

$$Q^{(2,1)}_{6;j_1k_1,j_2k_2,j_3m_1;\sigma_L=(13),\sigma_u=(12)} = Y^{(u)}_{j_1k_1} Y^{(u)}_{j_2k_2} Y^{(d)}_{j_3m_1} Y^{(u)*}_{j_3k_2} Y^{(u)*}_{j_2k_1} Y^{(d)*}_{j_1m_1} \tag{3.16}$$

However, (3.15) and (3.16) are not independent, since

$$Q^{(2,1)}_{6;j_1k_1,j_2k_2,j_3m_1;\sigma_L=(13),\sigma_u=(12)} = Q^{(2,1)}_{6;j_3k_1,j_1k_2,j_2m_1;\sigma_L=(23),\sigma_u=(12)} \tag{3.17}$$

Hence, as stated above, there are only two independent complex 6'th order invariants, which may be taken to be (3.15) and its mirror pair under $Y^{(u)} \leftrightarrow Y^{(d)}$. For brevity of notation, we shall denote these simply as

$$Q^{(fff')}_{j_1k_1,j_2k_2,j_3m_1} = Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f')}_{j_3m_1} Y^{(f)*}_{j_1k_2} Y^{(f)*}_{j_3k_1} Y^{(f')*}_{j_2m_1} \tag{3.18}$$

where $(fff') = (uud)$ or (ddu). Note the relations

$$P^{(f)}_{6;j_1k_1,j_2k_2,j_3k_3} = P^{(f)*}_{j_3k_2,j_2k_1,j_1k_3} \tag{3.19}$$

and

$$Q^{(fff')}_{6;j_1k_1,j_2k_2,j_3m_1} = Q^{(fff')*}_{6;j_1k_2,j_3k_1,j_2m_1} \tag{3.20}$$

So far, all of the results in this section hold for arbitrary N_G. For the physical case $N_G = 3$, the order 4 and order 6 invariants given above form a complete set, since, as was noted in Ref. [2] (for $N_G = 3$) there are are no new constraints from any invariant of order ≥ 8 (see below). We shall sometimes refer to P and Q invariants of order 4 and order 6 generically via the respective symbols P_4, Q_4, P_6, and Q_6.

We next review our theorems [2] and present the proofs. For a given model, construct the maximal set of N_{ia} independent complex invariants of lowest order(s), whose arguments (phases) are linearly independent. Then we have the

Theorem:

(a) each of these invariants implies a constraint that the elements contained within it cannot, in general, all be made simultaneously real; (b) this constitutes the complete set of constraints on which elements of $Y^{(u)}$ and $Y^{(d)}$ can be made simultaneously real; and hence (c) $N_p = N_{ia}$. These results hold for arbitrary N_G. Further, (d) for the physical case $N_G = 3$, the complex invariants of order 4 and 6 form a complete set, i.e., the argument of any complex invariant of order ≥ 8 can be expressed in terms of the arguments of complex invariants of orders 4 and 6, and hence these higher order complex invariants do not yield any new phase constraints. Hence also, for the physical case $N_G = 3$, the total number of independent (irreducible) complex invariants is given by

$$N_{inv} = N_{inv,4} + N_{inv,6}, \qquad for \quad N_G = 3 \tag{3.21}$$

(A discussion of complex invariants in the case of arbitrary N_G is given in section 4.)

Proof: The proof of (a) is essentially obvious: since by construction, the argument of each complex invariant is invariant under the rephasings of the fermion fields (2.5)-(2.7) and since it is not, in general, equal to 0 or π, the corresponding invariant cannot in general be made

real. It follows, *a fortiori*, that it is not, in general, possible to make the Yukawa matrix elements which comprise this complex invariant product all simultaneously real. Given that this is a maximal set, the assertions in parts (b) and (c) of the theorem follow immediately. The proof of (d) is more involved. Let X_{2n} be some non-zero invariant of type P or Q. Further, let $Y_{ij}^{(f)}$, $f = u$ or d, be one of the elements in X_{2n}. Then it is clear that in order for X_{2n} to be an invariant, it must also contain the product of $Y_{ik}^{(f')*} Y_{lj}^{(f)*}$ for some $k, l = 1, 2, 3$, where f' may be equal to, or different from, f. Hence we can write

$$X_{2n} = Y_{ij}^{(f)} Y_{lj}^{(f)*} Y_{ik}^{(f')*} X' \tag{3.22}$$

Now we consider the following two cases. (i) For some values of k and l, the Yukawa matrix element $Y_{lk}^{(f')}$ is non-zero. Then

$$\begin{aligned} \arg(X_{2n}) = \arg(Y_{lk}^{(f')*} Y_{lk}^{(f')}\, X_{2n}) = \\ \arg(Y_{lk}^{(f')*}\, (Y_{lk}^{(f')} Y_{ij}^{(f)} Y_{ik}^{(f')*} Y_{lj}^{(f)*})\, X') = arg(\tilde{X}_4\, \tilde{X}_{2n-2}) \end{aligned} \tag{3.23}$$

where

$$\tilde{X}_4 = Y_{lk}^{(f')} Y_{ij}^{(f)} Y_{ik}^{(f')*} Y_{lj}^{(f)*} \tag{3.24}$$

and

$$\tilde{X}_{2n-2} = Y_{lk}^{(f')} X' \tag{3.25}$$

Hence,

$$\arg(X_{2n}) = \arg(\tilde{X}_4) + \arg(\tilde{X}_{2n-2}) \tag{3.26}$$

Observe that eq. (3.24) subsumes both the case where $\tilde{X}_4$ is of P type and the case where it is of Q type. Given the definition (3.22), it is easily seen that (3.25) is an invariant. Therefore, in this case the argument of the order $2n$ invariant X_{2n} can be expressed as a sum of the arguments of an order 4 invariant $\tilde{X}_4$ and an order $(2n-2)$ invariant $\tilde{X}_{2n-2}$, so that $\arg(X_{2n})$ does not yield a new constraint. By induction, every invariant of order greater than 4 which satisfies the conditions of (i) is reducible in this manner.

It remains to consider the other case, that (ii) for all values of i, j, k and l, $Y_{lk}^{(f')} = 0$. In this case, the invariant X_{2n} cannot be reduced by means of the above procedure. Then it is

also true that since $Y_{lk}^{(f')}$ vanishes, it cannot be one of the elements in the X_{2n}, because the latter product is nonzero by assumption. Consequently, the invariant X_{2n}, which contains the product $Y_{ij}^{(f)}Y_{lj}^{(f)*}Y_{ik}^{(f')*}X'$, cannot contain the element $Y_{lk}^{(f')} = 0$ at the same time. Because each invariant contains either no elements, or at least two elements, from each column of the Yukawa mass matrices (in the latter case, denote the two as $Y_{pr}^{(f)}$ and $Y_{qr}^{(f)*}$), it follows that the X_{2n} invariant in this case must be constructed out of blocks $B_{pqr}^{(f)} = (Y_{pr}^{(f)}Y_{qr}^{(f)*})$ such that the pairs of the first indices, (p, q), of the two such blocks do not coincide:

$$X_{2n} = \prod_i B_{p_i\, q_i\, r}^{(f_i)}, \;\; where\ (p_i, q_i) \neq (p_j, q_j) \;\; if \;\; i \neq j \tag{3.27}$$

Since p and q can take only 3 values each, there are $\binom{3}{2} = 3$ possibilities to assign these indices (where $\binom{m}{n} = m!/(n!(m-n)!)$ denotes the binomial coefficient). This means that the order of an invariant which is not decomposable by these means cannot be greater than 3 times the order of B, i.e., 6. We can exhibit examples (see section 5) for which this bound is saturated, i.e. the order of the lowest-order nonvanishing complex invariant is 6. This thus proves (d). $\Box$

For most models the lowest-order nonvanishing complex invariants are of order 4, but, as mentioned, we will in section 5 exhibit some exceptions for which the lowest-order nonvanishing complex invariants are of order 6.

Corollary 1:

If in a given model there are as at least as many independent quartic complex invariants with independent arguments as there are unremovable phases, then the arguments of not only order 8 and higher, but also of 6'th order complex invariants are expressible in terms of those of the quartic invariants and hence yield no new phase constraints.

Proof:

This is obvious, since the premise of the corollary means that since all of the $N_p = N_{ia}$ unremovable phases have already been accounted for by quartic complex invariants, there cannot be any further phase constraints from any higher order complex invariants, and hence the arguments of all such higher order complex invariants must be expressible in terms of those of the N_{ia} quartic complex invariants with independent arguments. $\Box$

For most models, the lowest-order complex invariants are of order 4, but, as mentioned, we will exhibit two exceptions in section 5, where the only complex invariants are of order 6.

Corollary 2:

If in a given model all of the elements of the $Y^{(f)}$, $f = u, d$ are nonzero arbitrary (complex) quantities, then the arguments of the order 6, as well as the higher order, complex invariants are expressible in terms of those of the quartic invariants.

Proof

This is clear, since given any order 6 complex invariant, we can always apply the reduction method of eqs. (3.23)-(3.26) to express its argument in terms of that of the arguments of two quartic complex invariants. $\Box$

(Recall that in this case such a model would also have the maximal number of unremovable phases, (2.20).)

We have developed a graphical representation for the complex invariants which we have found useful. This is described next. The Yukawa matrix $Y^{(f)}$ is a $N_G \times N_G$ matrix whose rows and columns are labelled respectively by the first and second indices, j and k on the elements $Y^{(f)}_{jk}$. From the defining equation, (3.4), we can represent the quartic invariant $P^{(f)}_{j_1k_1,j_2k_2} = Y^{(f)}_{j_1k_1} Y^{(f)}_{j_2k_2} Y^{(f)*}_{j_2k_1} Y^{(f)*}_{j_1k_2}$ as follows. To the factor $Y^{(f)}_{j_1k_1} Y^{(f)*}_{j_2k_1}$ we associate an oriented line segment (which can be visualized as a vertical arrow $\uparrow$ or $\downarrow$ connecting the two indicated elements at the intersections of the rows j_1 and j_2 with the column k_1. Let the head of the arrow or equivalently the $\odot$ of the dipole be placed on the unconjugated element, and the tail of the arrow or equivalently the $\otimes$ of the dipole be placed on the conjugated element. For the figures, it is actually convenient to use a notation involving vertical dipoles $\substack{\odot \\ \otimes} = \uparrow$ and $\substack{\otimes \\ \odot} = \downarrow$, where, as indicated, the $\odot$ of the dipole corresponds to the head of the arrow (unconjugated element), while the $\otimes$ of the dipole corresponds to the tail of the arrow (conjugated element). To the factor $Y^{(f)}_{j_2k_2} Y^{(f)*}_{j_1k_2}$ we associate a similar oriented vertical line segment connecting the elements at the intersections of the j_2 and j_1 rows with the column k_2. Note that the head of this arrow points in the opposite direction from the first arrow, or equivalently, the dipole is oppositely oriented. These two opposite arrows link the same rows; consequently, they form a rectangle. Hence, we may associate with each complex quartic $P^{(f)}$ invariant a (possibly

square) rectangle in the matrix $Y^{(f)}$. As an example, our graphical representation of the quartic invariant $P^{(f)}_{22,33}$ is illustrated in Fig. 1(a).

Similarly, we can represent the quartic invariant $Q_{j_1k_1,j_2m_1} = Y^{(u)}_{j_1k_1}Y^{(d)}_{j_2m_1}Y^{(u)*}_{j_2k_1}Y^{(d)*}_{j_1m_1}$ as follows. Imagine the $Y^{(u)}$ matrix to lie next to the $Y^{(d)}$ matrix on a plane. To the factor $Y^{(u)}_{j_1k_1}Y^{(u)*}_{j_2k_1}$ we associate an oriented line segment linking the elements at the intersections of the j_1 and j_2 rows with the k_1'th column in the $Y^{(u)}$ matrix. As before, the head of the arrow is assigned to the unconjugated element. To the other factor $Y^{(d)}_{j_2m_1}Y^{(d)*}_{j_1m_1}$ we associate an arrow linking the elements at the intersections of the j_2 and j_1 rows with the m_1 column in the $Y^{(d)}$ matrix. As with P, the two arrows making up the Q point in opposite directions. We may thus associate with each complex quartic Q invariant a rectangle linking the arrow in the $Y^{(u)}$ matrix with the oppositely oriented arrow in the $Y^{(d)}$ matrix. As examples, the quartic invariants $Q_{12,22}$ and $Q_{23,32}$ are illustrated in Fig. 1(b,c). The operation of complex conjugation of a given complex quartic P or Q invariant reverses the directions of the arrows in each of the line segments. Since the complex conjugate invariants give the same phase constraints, the corresponding graphs with reversed arrows may be considered to be equivalent to the graphs already considered. Hence for purposes of enumerating invariants, one may suppress the degree of freedom associated with an overall reversal of all arrows (given that the pairs of columns forming each graph always have oppositely directed arrows), and thus deal with just the rectangles. This is symbolically indicated in Fig. 2 using the equivalent dipole notation.

Higher order invariants may be represented in a similar manner. A 6'th order P complex invariant in the sector f ($= u$ or d) is represented as three vertical arrows in three columns of $Y^{(f)}$ with the property that the head of each is matched by the tail of some other arrow. A 6'th order $Q^{(uud)}$ complex invariant is represented by two oppositely directed vertical arrows in $Y^{(u)}$ together with one vertical arrow in $Y^{(d)}$, again with the property that the head of each arrow is matched by the tail of some other arrow. Finally, a 6'th order $Q^{(ddu)}$ invariant is represented as for $Q^{(uud)}$ but with the interchange of $Y^{(u)}$ and $Y^{(d)}$. As examples, we show $Q^{(ddu)}_{32,23,12}$ and $P_{11,23,32}$ in Figs. Figs. 3(a) and 3(b).

Since the full set of N_{inv} independent complex invariants will have arguments which are

not, in general, linearly independent, it follows that

$$N_{inv} \geq N_{ia} \tag{3.28}$$

It may also happen that, e.g. for order $2n = 4$, the number of independent quartic complex invariants, $N_{inv,4}$ is greater than N_{ia}. For each complex invariant of a given order, X,

$$\arg(X) = \sum_{f=u,d} \sum_{j,k} c_{j,k}^{(f)} \arg(Y_{jk}^{(f)}) \tag{3.29}$$

where the sum is over the N_{eq} complex elements of $Y^{(u)}$ and $Y^{(d)}$. Including all orders of invariants, these equations can be written as

$$\xi = Zw \tag{3.30}$$

where ξ is the N_{inv}-dimensional vector

$$\xi = (\arg(X_1), ..., \arg(X_{N_{inv}}))^T \tag{3.31}$$

Z is an N_{inv}-row by N_{eq}-column matrix, and it is implicitly understood here that all of the $\eta_{jk}^{(f)}$ in eq. (2.11) for w are taken to be zero. Often, some nonzero elements of $Y^{(u)}$ and $Y^{(d)}$ do not occur in any complex invariants and hence do not occur on the right-hand side of (3.29). These elements may be rephased freely. The Z matrix as defined in eq. (3.30) has columns of zeroes corresponding to each of these elements. For simplicity, one may thus define a reduced vector w_r whose dimension is equal to the number of elements of $Y^{(f)}$, $f = u, d$, which occur in complex invariants, and a corresponding reduced matrix Z_r defined by

$$\xi = Z_r w_r \tag{3.32}$$

Clearly, by removing columns of zeroes we do not reduce the rank, so

$$rank(Z_r) = rank(Z) \tag{3.33}$$

Then the number of independent arguments among the complex invariants is given by the rank of Z:

$$N_{ia} = rank(Z) \tag{3.34}$$

The eqs. (3.30) and (3.32) subsume the complex invariants of all orders (i.e. for the physical case $N_G = 3$, orders 4 and 6). One can also apply this method to determine the linearly independent invariants of a given order, $2n$. For this purpose, we define a vector of complex invariants of this order:

$$\xi_{2n} = (\arg(X_{2n;1}), ..., \arg(X_{N_{inv,2n}}))^T \tag{3.35}$$

We then have

$$\xi_{2n} = Z_{2n} w \tag{3.36}$$

As before, one can define a reduced vector $w_{r,2n}$ composed of the elements of $Y^{(u)}$ and $Y^{(d)}$ which occur in the complex invariants of order $2n$, and a corresponding reduced matrix $Z_{r,2n}$ satisfying

$$\xi_{2n} = Z_{r,2n} w_{r,2n} \tag{3.37}$$

Then the number of linearly independent arguments among the complex invariants of order $2n$ is given by $rank(Z_{2n}) = rank(Z_{r,2n})$.

4 Further Results for General N_G

In this section we give a number of new results on complex invariants and unremovable phases in Yukawa matrices for an arbitrary number of fermion generations, N_G. A consideration of general N_G yields further insight into the physical case $N_G = 3$. However, the results in this section will not be used directly in our analysis of realistic models, so that it may be skipped by the reader who is mainly interested in phenomenological applications.

It is interesting to consider the case of Yukawa matrices with all arbitrary nonzero elements, as we did before, in the context of our first theorem on the number of unremovable phases. Just as this case clearly yields the maximal number of unremovable phases, so also it yields the maximal number of complex invariants. We have proved above that in Corollary 2 that in this case the complex quartic invariants form a complete set, in the sense that the arguments of the order $2n$ invariants for $2n \geq 6$ can be expressed in terms of those of the quartic invariants.

We now calculate the maximal number of (independent) complex quartic invariants. This is done, as explained above, by performing the calculation in the case where the $Y^{(f)}$ have all nonzero arbitrary (complex) elements, since this case clearly yields the maximal number of such invariants. Our first result is that the maximal number of complex quartic $P^{(f)}$ invariants of each type f ($= u$ or d) is

$$(N_{P_4,f})_{max} = \binom{N_G}{2}^2 \tag{4.1}$$

Proof:
For a given $f = u$ or d, the quartic $P^{(f)}$ invariants are obtained by choosing all rectangles (some of which will be square). To pick a rectangle, we pick any two different columns of $Y^{(f)}$; this can be done in $\binom{N_G}{2}$ ways. (Note that if we were to choose the two columns to coincide, we would get a doubly occupied line segment, corresponding to a real invariant.) These columns are equivalent (whence the division by 2!) because of the fact, noted above, that since each complex invariant and its complex conjugate yield the same phase constraint, we can count rectangles of the form $\begin{smallmatrix}\odot\otimes\\ \otimes\odot\end{smallmatrix} =\uparrow\downarrow$ and $\begin{smallmatrix}\otimes\odot\\ \odot\otimes\end{smallmatrix} =\downarrow\uparrow$ as equivalent. Within one of the columns we pick any two different elements; this can be done in any of $\binom{N_G}{2}$ ways. Therefore, the number of (independent) complex quartic $P^{(f)}$ invariants of each type f for this case, which is the maximal number of such invariants, is as given in (4.1). $\square$

Since the maximal number of quartic $P^{(f)}$ complex invariants of each type $f = u$ or d, $N_{P_4,f,max}$, is the same for $f = u$ and $f = d$, the total maximal number of quartic P complex invariants is just double the right-hand side of (2.20), i.e.

$$(N_{P_4})_{max} = \frac{[N_G(N_G-1)]^2}{2} \tag{4.2}$$

For the maximal number of complex quartic Q invariants, we find

$$(N_{Q_4})_{max} = \frac{N_G^3(N_G-1)}{2} \tag{4.3}$$

Proof:
As above, the maximal value is calculated by considering the case where $Y^{(u)}$ and $Y^{(d)}$ have all nonzero arbitrary complex elements. First, we pick any column in, say, $Y^{(u)}$; this can be

done in N_G ways. Within this column we choose two different elements, which can be done in $\binom{N_G}{2}$ ways. This determines one side of the rectangle. To pick the other side, we choose any column of $Y^{(d)}$; this can be done in N_G ways. The two elements within this column are, of course, already determined by those in the column of $Y^{(u)}$. As before, the cases $\uparrow\downarrow$ and $\downarrow\uparrow$ give complex conjugate invariants and hence the same phase constraint, so that one does not have to treat them together, which is equivalent to ignoring an overall reversal in the directions of all arrows. Hence, the number of complex quartic Q invariants is $N_G\binom{N_G}{2}N_G$, i.e. the result given in (4.1). $\square$

Hence, finally, the maximal value of the total number of independent complex quartic invariants is

$$\begin{aligned}(N_{inv,4})_{max} &= (N_{P_4})_{max} + (N_{Q_4})_{max} \\ &= \frac{N_G^2(N_G-1)(2N_G-1)}{2}\end{aligned} \tag{4.4}$$

For the specific cases N_G =1, 2, and 3, we thus have

$$((N_{P_4})_{max},\ (N_{Q_4})_{max},\ (N_{inv,4})_{max}) = (0,0,0),\ (2,4,6),\ (18,27,45) \tag{4.5}$$

Continuing with the analysis of this case in which all of the elements of $Y^{(f)}$, $f = u, d$ are nonzero arbitrary complex quantities, we immediately see that for this case (except for $N_G = 1$ for which $(N_{inv})_{max} = (N_p)_{max} = 0$) the resultant maximal number of unremovable phases, or, by our theorem on invariants, equivalently, the maximal number of independent arguments among the complex invariants, $(N_{ia})_{max} = (N_p)_{max}$, is always less than $(N_{inv,4})_{max}$, so that the arguments of the various complex quartic invariants are not all independent. This is proved easily by recalling our earlier result (2.20), which, with (2.17), yields

$$(N_{ia})_{max} = (N_p)_{max} = (N_G-1)(2N_G-1) \tag{4.6}$$

Comparison with our result for $(N_{inv,4})_{max}$ in eq. (4.4) shows that

$$(N_{inv,4})_{max} > (N_{ia})_{max} \quad for\ N_G \geq 2 \tag{4.7}$$

We further recall from Corollary 2 to our theorem on invariants that in this case where all elements of the $Y^{(f)}$ are nonzero arbitrary complex numbers, the arguments of the 6'th and higher order complex invariants can be expressed in terms of those of the quartic invariants. Although in this case the 6'th order invariants do not yield any further phase constraints, it is of interest to calculate the maximal number of these invariants since it shows how this number depends on N_G.

For the number of independent 6'th order complex P invariants in this case of $Y^{(f)}$ with all nonzero arbitrary (complex) entries, which gives the maximal number of complex P_6 invariants, we get

$$(N_{P_6,f})_{max} = \frac{[N_G(N_G-1)(N_G-2)]^2}{3!} \tag{4.8}$$

Proof:

We calculate this as follows. First, we pick a column, in $Y^{(f)}$ ($f = u$ or d), which can be done in N_G ways. Within this column, we pick two different elements, which can be done in $\binom{N_G}{2}$ ways. Next, we pick another column in this matrix, which can be done in $(N_G - 1)$ ways. In this column, we pick the element which is at the same level (i.e. has the same value of the first index) as the head of the arrow in the first column, or the element which is at the same level as the tail of this arrow; there is a factor of 2 corresponding to these two choices. We then pick a second element in this column, which is different from the first and is also not at the same level of the other end of the arrow in the first column (since if it were, we could immediately form a complex quartic invariant, i.e. we would not be constructing a genuine 6'th order complex invariant). There are $(N_G - 2)$ ways of choosing this second element. Finally, we choose the third column, which is different from the first two; this can be done in $(N_G - 2)$ ways. The vertical levels of the elements in this column are now automatically determined from the previous choices. Since all of the columns are equivalent, we divide by 3!. Putting these factors together gives, for the number of complex 6'th order P invariants, the result $N_G\binom{N_G}{2}(N_G-1)\cdot 2\cdot(N_G-2)(N_G-2)/3!$, which yields (4.8). $\square$

Including the $f = u$ and $f = d$ sectors, one obtains for the maximal value of the total number of P_6 complex invariants just doubles the right-hand side of (4.8).

For the maximal number of independent 6'th order complex $Q^{(fff')}$ invariants, we calcu-

late

$$(N_{Q6,fff'})_{max} = \frac{N_G^3(N_G-1)^2(N_G-2)}{2} \tag{4.9}$$

Proof:

We consider the $Y^{(f)}$ matrix first. As before, we pick a column, and two different elements within this column, which can be done in $N_G\binom{N_G}{2}$ ways. We then pick a different column within this matrix; this can be done in (N_G-1) ways. In this column we pick the element which is at the same level (i.e. has the same value of the first index) as the head of the arrow in the first column, or the element which is at the same level as the tail of this arrow; there is a factor of 2 corresponding to these two choices. We then pick a second element in this column, which is different from the first and is also not at the same level of the other end of the arrow in the first column (since if it were, we could immediately form a complex quartic invariant, i.e. we would not be constructing a genuine 6'th order complex invariant). There are (N_G-2) ways of choosing this second element. These two columns are equivalent, so we divide by a factor of 2!. Finally, we pick a column in the other matrix, $Y^{(f')}$, which can be done in N_G ways. The elements in this column are now completely determined by the previous choices. This yields the result $N_G\binom{N_G}{2}(N_G-1)\cdot 2\cdot(N_G-2)\cdot(2!)^{-1}\cdot N_G$, i.e. the number in (4.9). $\Box$

Including both the $fff' = uud$ and ddu terms just doubles the number in (4.9). Hence, for the total number of independent complex 6'th order invariants in this case, we finally get

$$\begin{aligned}(N_{inv,6})_{max} &= 2[(N_{P6,f})_{max} + (N_{Q6,fff'})_{max}] \\ &= \frac{2N_G^2(N_G-1)^2(N_G-2)(2N_G-1)}{3}\end{aligned} \tag{4.10}$$

Of course, in the case where the $Y^{(f)}$ have all nonzero arbitrary complex elements, we have already shown that the quartic complex invariants form a complete set, so that these 6'th order invariants do not yield any new phase constraints. One use of the formulas (4.8)-(4.10) is that they show that there are no complex 6'th order invariants for $N_G = 2$. We will show below that all higher order complex invariants also vanish for $N_G = 2$.

The maximal number of independent complex invariants of order ≥ 8 may be calculated in a similar manner. Here, as explained before, we consider only irreducible 8'th order invariants, i.e. those which cannot be factored into products of lower-order invariants. Thus, for example, 8'th order invariants formed as the products of two different quartic invariants are not counted. Since the proofs become rather long, we will simply give the results. For the maximal number of (irreducible) 8'th order $P^{(f)}$ invariants of each type $f = u$ or d we find

$$(N_{P_8,f})_{max} = \frac{N_G^2(N_G-1)^2(N_G-2)(N_G-3)^3}{3!} \tag{4.11}$$

For the maximal number of complex $Q_8^{(3,1)}$ invariants (which is, of course, equal to that of the $Q_8^{(1,3)}$ invariants), we calculate

$$(N_{Q_8^{(3,1)}})_{max} = (N_{Q_8^{(1,3)}})_{max} = \frac{N_G^3(N_G-1)^2(N_G-2)(N_G-3)^2}{3} \tag{4.12}$$

Finally, for the maximal number of complex $Q_8^{(2,2)}$ invariants, we get

$$(N_{Q_8^{(2,2)}})_{max} = \frac{N_G^3(N_G-1)^3(N_G-2)(N_G-3)}{4} \tag{4.13}$$

We now observe that the maximal numbers of each of these various types of 8'th order invariants, $(N_{P_8,f})_{max}$ in (4.11), $(N_{Q_8^{(3,1)}})_{max} = (N_{Q_8^{(1,3)}})_{max}$ in (4.12), and $(N_{Q_8^{(2,2)}})_{max}$ in (4.13), all vanish unless $N_G \geq 4$. This explicit calculation thus provides an illustration of the general result in part (d) of our theorem on invariants. Recall that that result applied more generally to all complex invariants, including reducible ones, and stated that for the physical case $N_G = 3$ the arguments of any complex invariant of order ≥ 8 could be expressed in terms of the arguments of complex invariants of order 4 and 6. Our general calculation above shows that, for the physical case $N_G = 3$, there are no irreducible complex invariants of order 8, i.e. all 8'th order complex invariants are products of lower-order invariants, and hence obviously their arguments can be expressed in terms of those of these lower-order complex invariants.

One might be curious how part (d) would read if N_G were equal to 2. The answer is:

Theorem:

For $N_G = 2$, the only (irreducible) complex invariants are of 4'th order.

Proof:
To prove this, we calculate the maximal number of complex invariants as before, by using the case where the $Y^{(f)}$ have all nonzero arbitrary complex elements. Consider first a $P^{(f)}_{2n}$ complex invariant of order $2n \geq 6$. To form this, one must pick $n \geq 3$ different columns from $Y^{(f)}$. But this is impossible, since $Y^{(f)}$ has only $N_G = 2$ columns. Consider next a 6'th order complex Q-type invariant, say $Q^{(2,1)_6}$. To form this, one must pick two columns from $Y^{(u)}$ with the property that the ends of the arrows do not match each other (if they did, one would form a P_4 invariant). But this is impossible since $Y^{(u)}$ is just a 2×2 matrix. Similar reasoning shows that it is also impossible to form any higher-order irreducible complex invariant. □
Of course, there may be reducible complex invariants of higher order, e.g., products of two different quartic invariants.

For this case $N_G = 2$ it is easy to construct all such invariants. To obtain the maximal set, we assume that the $Y^{(f)}$ have all nonzero arbitrary (complex) elements; obviously, if some elements are zero, the resultant set of complex invariants would be commensurately reduced. According to our general result (4.1), $(N_{P_4,f})_{max} = 1$, and the explicit complex invariants are

$$P^{(f)}_{11,22} = Y^{(f)}_{11} Y^{(f)}_{22} Y^{(f)*}_{21} Y^{(f)*}_{12} \tag{4.14}$$

for $f = u, d$. From our general result (4.3) it follows that there are $(N_{Q_4})_{max} = 4$ independent complex quartic Q invariants. These are

$$Q_{11,22} = Y^{(u)}_{11} Y^{(d)}_{22} Y^{(u)*}_{21} Y^{(d)*}_{12} \tag{4.15}$$

$$Q_{12,21} = Y^{(u)}_{12} Y^{(d)}_{21} Y^{(u)*}_{22} Y^{(d)*}_{11} \tag{4.16}$$

$$Q_{11,21} = Y^{(u)}_{11} Y^{(d)}_{21} Y^{(u)*}_{21} Y^{(d)*}_{11} \tag{4.17}$$

and

$$Q_{12,22} = Y^{(u)}_{12} Y^{(d)}_{22} Y^{(u)*}_{22} Y^{(d)*}_{12} \tag{4.18}$$

From eq. (2.20), we know that there are $N_p = 3$ unremovable phases for this case, and correspondingly, among the $N_{inv} = 6$ complex (quartic) invariants there are $N_{ia} = N_p = 3$

invariants with linearly independent arguments. These may, for example, be taken to be $P^{(u)}_{11,22}$, $P^{(d)}_{11,22}$, and $Q_{11,22}$.

Returning to the case of general N_G, as one makes some of the elements of the $Y^{(u)}$ and $Y^{(d)}$ zero, the number of unremovable phases and the number of complex invariants are decreased. The amount by which the number of unremovable phases is decreased is not solely a function of the number of zeroes, but in addition depends on their locations. For almost all of the models which we have studied, one finds a simple empirical relation $N_p = (N_p)_{max} - N_z = (2N_G - 1)(N_G - 1) - N_z$, where we recall that N_z is the total number of zero elements in $Y^{(u)}$ and $Y^{(d)}$. This is not a general result; in section 5 we exhibit a toy model (no. 8) which, considered as an $N_G = 3$ model, is an exception. However, that model is degenerate, in the sense that it involves a decoupled third generation in both up and down quark sectors. If one considers only the mutually coupled sector of the model, then it reduces to a $N_G = 2$ model, and does obey the $N_G = 2$ version of the above empirical relation.

As a final item in this section, we contrast the unremovable and hence physically meaningful phases in the quark Yukawa (or mass) matrices, with the unremovable phase(s) in the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix, V, defined by the charged weak current

$$J^\mu = \bar{u}_{jL,m} \gamma^\mu V_{jk} d_{kL,m} \tag{4.19}$$

where $u_{jL,m}$ and $d_{kL,m}$ are the left-handed components of the mass eigenstates of the $Q = 2/3$ and $Q = -1/3$ quark fields (indicated by the subscript m) defined after the spontaneous symmetry breaking of $SU(2) \times U(1)$, with $u_{1L,m} = u_{L,m}$, $u_{2L,m} = c_m$, $u_{3L,m} = t_{L,m}$, and similarly for $d_{jL,m}$. If the mass matrices are diagonalized by the biunitary transformations $U_{f,L} Y^{(f)} U^\dagger_{f,R} = Y^{(f)}_{diag}$ for $f = u, d$, then $V = U_{u,L} U^\dagger_{d,L}$. The rephasing properties of V are quite different from those of the $Y^{(f)}$, as is obvious from the fact that the charged weak current is chirality-preserving, connecting left-handed with left-handed chiral components of individual physical mass eigenstates of quark fields, whereas the Yukawa terms connect left-handed with right-handed chiral components. Furthermore, the Yukawa matrices, being defined via $SU(3) \times SU(2) \times U(1)$-invariant Yukawa couplings, involve the actual $SU(3) \times SU(2) \times U(1)$

fields, i.e. the N_G left-handed SU(2) doublets Q_{jL} and the $2N_G$ right-handed SU(2) singlets u_{jR} and d_{jR} (not mass eigenstates). The rephasing equations for the Yukawa matrices were given above in (2.5)-(2.7). In contrast, when one determines the unremovable phases in V, since it is defined in terms of the mass eigenstates themselves, the rephasing equations which one uses are

$$\begin{aligned} u_{j,Lm} &= e^{i\theta_j} u'_{j,Lm} \\ d_{j,Lm} &= e^{i\phi_j} d'_{j,Lm} \end{aligned} \tag{4.20}$$

for $j = 1, ..., N_G$. The number of these rephasing equations is $2N_G$, as opposed to the $3N_G$ rephasing equations (2.5)-(2.7) for the Yukawa matrices. The resultant CKM matrix is

$$V'_{jk} = e^{i(-\theta_j+\phi_k)} V_{jk} \tag{4.21}$$

Recall the well-known argument that since (i) V is unitary, and hence specified by N_G^2 real parameters, (ii) there are $2N_G$ rephasings as noted; and (iii) the overall rephasing U(1) defined by (4.20) with $\theta_j = \phi_j$ for all $j = 1, ..., N_G$ leaves V invariant, it follows that there are $N_G^2 - (2N_G - 1) = (N_G - 1)^2$ real parameters specifying V, of which $N_G(N_G + 1)/2$ are rotation angles and the remainder are

$$N_{p,V} = \frac{(N_G - 1)(N_G - 2)}{2} \tag{4.22}$$

unremovable, physically meaningful, phases. An immediate contrast is that in general, $N_{p,V}$ is only a function of N_G, whereas the number of unremovable phases in the Yukawa matrices do not just depend on N_G but rather on the detailed structure of these matrices. An important inequality is that the number of unremovable phases in the quark Yukawa matrices is at least as large as the number of unremovable phases in the CKM matrix V:

$$N_p \geq N_{p,V} \tag{4.23}$$

The proof of this is obvious since the phases in V originated, as V itself did, from the diagonalization of the Yukawa (equivalently, mass) matrices as defined in (2.1). Our discussion

of specific realistic models will illustrate this inequality; in particular, in the physically interesting case $N_G = 3$, the CKM matrix has $N_{p,V} = 1$ unremovable (CP-violating) phase, whereas the realistic models typically have at least $N_p = 2$ unremovable phases in the Yukawa matrices. From our inequality (4.23), there follows the theorem that phases in quark mass matrices do not necessarily violate CP. The issue of CP violation involving Yukawa couplings also depends on how many Higgs are present in the theory and how they couple to the quarks. In order to make our analysis as general as possible, we have defined the mass and Yukawa matrices as in eq. (2.1) without explicitly specifying the precise Higgs content of the theory; however, most recent studies of actual models (including our own) assume the minimal supersymmetric standard model, for which the Higgs were listed in (2.3) and (2.4). For a typical case where $N_p > N_{p,V}$, i.e. $N_p > 1$ for the $N_G = 3$ case of physical interest, it follows that some phases in the quark mass matrices may contribute to CP-conserving quantities. In the mathematical case $N_G = 1$, eqs. (2.20) and (4.22) imply that $(N_p)_{max} = N_{p,V} = 0$. For the case $N_G = 2$, the CKM matrix still has no phase, but (2.20) shows that N_p may be as large as 3. Indeed, in the specific $N_G = 2$ case worked out completely in eqs. (4.14)-(4.18), N_p is equal to 3. Thus, in this $N_G = 2$ case, the phases in the Yukawa (mass) matrices do not yield any phase in the CKM matrix.

5 Applications to Specific Models

Although our theorems are quite general, it is useful to see how they apply to various specific models. We shall do this in the present section. Over the years, a large number of models have been proposed for the origin of quark masses and mixing. In some of the earliest efforts, the forms of the models were hypothesized to apply near to the electroweak mass scale. In general, various studies can be classified according to whether they assume a theoretical framework of perturbative or nonperturbative electroweak symmetry breaking. We shall concentrate here on the class of models in which the observed electroweak symmetry breaking is perturbative. In this class, an appealing framework is provided by supersymmetric extensions of the standard model, which stabilize the Higgs sector and hence stabilize

the hierarchy between the electroweak scale and the Planck scale. Among these supersymmetric extensions, the minimal supersymmetric standard model (MSSM) has the particular feature that it yields gauge coupling unification, thereby providing a self-consistent basis for a supersymmetric grand unified theory (SUSY GUT) [5]. In turn, for studies of fermion masses and mixing, a (supersymmetric) grand unified theory has the advantages that (i) one can naturally relate quark and lepton masses; (ii) with the grand unified group SO(10) one can naturally obtain (complex) symmetric Yukawa matrices at the GUT scale, which helps to minimize parameters. Thus, a number of studies have been carried out of particular (complex) symmetric forms for Yukawa matrices at the (supersymmetric) grand unified scale, in which the renormalization group equations of the MSSM are used to evolve these forms down to the electroweak scale, where they are diagonalized to yield the known quark masses and the CKM quark mixing matrix V. In these studies, one assumes some symmetries (which are usually discrete in current work) to prevent various Yukawa couplings and thereby render various entries in the Yukawa matrices zero. The purpose of this is, of course, to minimize the number of parameters and hence increase the apparent predictiveness. (Parenthetically, we recall that when one evolves these forms for the Yukawa matrices down to the electroweak level, they do not, in general, remain precisely symmetric, and some of the zero entries can become nonzero.) These studies are phenomenological in the sense that they do not explain the origin of these discrete symmetries. The most likely origin of the discrete symmetries is probably an underlying string theory (in which these are actually local symmetries). However, ironically, string theories do not generically yield either simple gauge groups (the essence of grand unification) or symmetric (complex or real) mass matrices, at least at the present limited level of understanding of their physical properties [6]. At a phenomenological level it is reasonable to seek forms for the $Y^{(f)}$ with as many zero entries as possible. However, if and when one understands fermion masses and mixing in terms of a truly fundamental theory, with perhaps no free parameters, then since everything is (at least in principle) calculable, it makes no difference whether the elements of the $Y^{(f)}$ are exactly zero or just small. The entries which are modelled as being exactly zero might well be nonzero, calculable quantities suppressed by sufficiently high powers of $(M_r/\bar{M}_P)$, where

M_r is the reference scale at which one analyzes the effective Yukawa interaction. Thus, the real goal is not necessarily to maximize the number of zero entries in the Yukawa matrices of a viable model as one necessarily tries to in a phenomenological approach; rather, it is to gain an understanding of the (presumably unique) fundamental theory, together with a calculational ability which would enable one to calculate these matrices from first principles. Given the widely acknowledged [6] limitations on one's understanding of the phenomenological consequences of strings and, at the level of this understanding, the apparent lack of uniqueness of string theory, as evident, e.g., in the 4D free-fermion constructions, it does not appear that one is very close to achieving this goal at present. Finally, for completeness, we must note that there are a number of concerns which must be faced in actual model-building with supersymmetric theories, such as, for example, the "μ problem" and the problem of splitting light Higgs doublets from other components of grand unified Higgs representations which must be heavy, and so forth; these are reviewed, e.g., in [5].

With these brief theoretical comments, we proceed to illustrate our general theorems with specific models. We emphasize that our purpose here is not to propose new viable models of quark Yukawa matrices with the attendant renormalization group calculations and comparison with experiment; rather, our purpose is to consider (generalizations of) forms which are known to be experimentally viable and, for these, to determine the unremovable phases and which elements can be made real by rephasings. Since our results do not depend on whether or not the Yukawa matrices are symmetric at some mass scale, we shall consider the general case of non-symmetric Yukawa matrices.[8] As a basis for this, we will use a valuable recent study of SUSY GUT Yukawa matrices with the restriction that $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$ [7]. This work presented an exhaustive list of five viable models, with this restriction, which we label M_j, $j = 1-5$. For the reasons mentioned above, we shall actually consider the generalizations of these models in which $|Y^{(f)}_{jk}|$ is not necessarily equal to $|Y^{(f)}_{kj}|$. Again, we emphasize that this does not affect the counting of the number of unremovable phases or the determination

[8] From a SUSY GUT model-builder's point of view, this might seem perverse, since symmetric Yukawa matrices have the appeal of reducing the number of parameters and, moreover, can be naturally obtained in grand unified theories. But this appeal is offset by one's awareness of the results of string-based studies, mentioned above, which suggest that nature may not, in fact, be described by either a grand unified group or by symmetric Yukawa matrices.

of which elements can be made real by rephasing. We label our five generalizations as M'_j, $j = 1-5$ and discuss them in the next five subsections. Since the respective special cases are experimentally viable, it follows, *a fortiori*, that the generalizations are also viable, i.e., for appropriate choices of the parameters, one finds predictions in agreement with experimental constraints on the CKM matrix for experimentally acceptable input values of the top quark mass. Our set of models is not intended to be exhaustive, since a comprehensive study of viable non-symmetric Yukawa matrices has not been performed (for the understandable reason that at a phenomenological level such models have too many parameters to make useful predictions). Model 6 is related to a model with non-symmetric Yukawa matrices recently sketched in a string context[8]. Models 7 and 8 are toy examples to illustrate certain theoretical points.

5.1 Model 1

The first illustrative model is defined by the Yukawa matrices

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \tag{5.1.1}$$

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \tag{5.1.2}$$

where here, and below, each element in the Yukawa matrices is, in general, complex, since no symmetry requires it to be real. This model has $N_{z,u} = 5$, $N_{z,d} = 3$, $N_z = 8$, and $N_{eq} = 10$. In the physical context, the forms in eqs. (5.1.1) and (5.1.2) are assumed to hold at a given mass scale, such as the SUSY GUT mass scale $M_{GUT} = (1-2) \times 10^{16}$ GeV where the gauge couplings coincide in the simple MSSM calculation, or a higher mass scale near to the (reduced) Planck mass $\bar{M}_P = 2.4 \times 10^{18}$ GeV, which would be expected in supergravity and string theories, in which one would extend the MSSM calculation to take account of string threshold corrections and additional fields which could be present. Special cases of this model have been studied in Refs. [7] and [9]. Besides the assumption made in both of these works, that $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$, $f = u, d$, Ref. [9] also took $|B_{22}| = |B_{23}|$.

We will give a detailed, pedagogical discussion of the application of our methods for this model. We first calculate the T matrix. The reader may recall here the definition in eq. (2.13) and the fact that the ordering of the columns of T is determined by the definition of the vector v in (2.10) while the ordering of the equations in (2.9), and hence the ordering of the elements of the vector w and the rows of T, is defined by the sequence of nonzero elements $Y_{jk}^{(f)}$ in (2.11) with $jk = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}$ for $f = u$ and then $f = d$. Explicitly, here the rows correspond to the rephasing equations for $Y_{12}^{(u)}$, $Y_{21}^{(u)}$, $Y_{22}^{(u)}$, $Y_{33}^{(u)}$, $Y_{12}^{(d)}$, $Y_{21}^{(d)}$, $Y_{22}^{(d)}$, $Y_{23}^{(d)}$, $Y_{32}^{(d)}$, and $Y_{33}^{(d)}$. We obtain the $N_{eq} \times 3N_G = 10$-row by 9-column matrix

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.1.3}$$

where the subscript refers to the model. Further, we calculate the rank of this matrix, $rank(T_1) = 8$. We can then apply our theorem in (2.17) to conclude that this model has $N_p = N_{eq} - rank(T) = 2$ unremovable, and hence physically meaningful, phases in the Yukawa matrices $Y^{(f)}$ (and thus of course also in the mass matrices). According to our theorem on invariants, there are therefore $N_{ia} = N_p = 2$ complex invariants with independent arguments. For this model we find that $N_{inv,4} = N_{ia}$, i.e. the number of quartic complex invariants is equal to the number of complex invariants with independent arguments, so that the two complex quartic invariants fully account for the two unremovable phases and corresponding phase constraints. Explicitly, we find the (nonzero, independent) complex quartic invariants

$$\begin{aligned} P_{22,33}^{(d)} &= Y_{22}^{(d)} Y_{33}^{(d)} Y_{32}^{(f)*} Y_{23}^{(f)*} \\ &= B_{22} B_{33} B_{32}^{*} B_{23}^{*} \end{aligned} \tag{5.1.4}$$

and

$$Q_{12,22} = Y^{(u)}_{12} Y^{(d)}_{22} Y^{(u)*}_{22} Y^{(d)*}_{12}$$
$$= A_{12} B_{22} A^*_{22} B^*_{12} \tag{5.1.5}$$

The graphical representations of these complex invariants were given above in Fig. 1(a) and Fig. 1(b), respectively. Associated with these are their two rephasing-invariant arguments (phases), $\arg(P^{(d)}_{22,33})$ and $\arg(Q_{12,22})$.[9] These complex invariants and associated invariant arguments imply constraints on which elements of $Y^{(u)}$ and $Y^{(d)}$ can be made real. Since, in general, $\arg(P^{(d)}_{22,33}) \neq 0, \pi$, it follows that (i) at least one of the $N_p = 2$ unremovable phases must reside among the set

$$S_{P^{(d)}_{22,33}} = \{B_{22}, B_{23}, B_{32}, B_{33}\} \tag{5.1.6}$$

and (ii) the 2×2 submatrix in $Y^{(d)}$ formed by the set (5.1.6),

$$\begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{33} \end{pmatrix} \tag{5.1.7}$$

and hence also $Y^{(d)}$ itself, cannot be made real or hermitian.[10] Second, since, in general, $\arg(Q_{12,22}) \neq 0, \pi$, it follows that the elements in the set

$$S_{Q_{12,22}} = \{A_{12}, A_{22}, B_{12}, B_{22}\} \tag{5.1.8}$$

(iii) cannot in general be made simultaneously real and thus, (iv) if one chooses $Y^{(u)}$ real, then it is not possible to make B_{12} and B_{22} both real. These constitute the complete set

[9] In the model of Ref. [2] the phase $\theta_{Q_{12,22}} = \arg(Q_{12,22})$ did not significantly help the fit to experimental data and hence was taken to be zero to minimize the number of parameters.

[10] As noted in Ref. [2], our phase results disagree with those in Ref. [7] for the five models which they consider. In particular, we disagree with (i) their statement (footnote on p. 21 of Ref. [7]) that (complex) 3×3 symmetric mass or Yukawa matrices can be transformed to hermitian form by rephasings of the quark fields; (ii) the counting of unremovable phases in the Yukawa matrices in that work; and (iii) the claims made concerning which elements of these matrices can be made real by rephasings of fermion fields. For the models M'_j, $j = 1 - 5$ considered here, and hence also for the corresponding models M_j of Ref. [7], we find the values of N_p are respectively $2, 3, 2, 2, 2$, not $1, 2, 1, 1, 1$ as claimed there. In Ref. [7] the rephased forms of the Yukawa matrices (their Table 1) list $Y^{(d)}$ as being hermitian and $Y^{(u)}$ as being real for all five models. We find that $Y^{(d)}$ cannot in general be made hermitian (nor can the submatrix (5.1.7) be made real or hermitian) in models 1, 2, and 3. We also find that $Y^{(u)}$ cannot be made real (or hermitian) in models 4 and 5. We have communicated our results to P. Ramond and G. G. Ross and thank them for discussions. Our differences do not invalidate the important conclusions of Ref. [7] that these five forms are experimentally viable, since the inclusion of more phases can only improve the fit to experiment.

of rephasing constraints on the $Y^{(f)}$, $f = u, d$. These constraints allow both phases to be put in $Y^{(d)}$ and both to be put in the set $S_{P^{(d)}_{22,33}}$. If one chooses to make B_{22} and B_{12} real, then one cannot make $Y^{(u)}$ real, and must assign one phase to A_{12} or A_{22} and the second to B_{23}, B_{32} or B_{33}. The elements A_{21}, A_{33}, and B_{21} do not occur in any complex invariants; consequently, their phases are unconstrained and may be rephased to arbitrary values by the rephasing transformations (2.5)-(2.7). Our results show that it is not, in general, true that a (complex) symmetric Yukawa or mass matrix can be transformed to a hermitian matrix by the rephasings of fermion fields.

As an explicit illustration, after rephasings, one could obtain

$$Y^{(u)'} = \begin{pmatrix} 0 & |A_{12}| & 0 \\ |A_{21}| & |A_{22}| & 0 \\ 0 & 0 & |A_{33}| \end{pmatrix} \tag{5.1.9}$$

and

$$Y^{(d)'} = \begin{pmatrix} 0 & |B_{12}|e^{-i\ \arg(Q_{12,22})} & 0 \\ |B_{21}| & |B_{22}|e^{i\ \arg(P^{(d)}_{22,33})} & |B_{23}| \\ 0 & |B_{32}| & |B_{33}| \end{pmatrix} \tag{5.1.10}$$

where for any real quantity, $|Y^{(f)}_{jk}|$, one could have obtained $-|Y^{(f)}_{jk}|$ due to the freedom of using $\eta^{(f)}_{jk} = 1$ rather than $\eta^{(f)}_{jk} = 0$ in (2.9). Since $\arg(Y^{(d)}_{21}) = \arg(B_{21})$ is unconstrained, one could also have obtained $Y^{(d)'}_{21} = |B_{21}|e^{i\ \arg(Q_{12,22})}$ in eq. (5.1.10), as discussed in connection with eq. (2.18), although of course this choice has no effect on the physics. An example of a form which could not, in general, be obtained from any rephasings of fermion fields is the hermitian form

$$Y^{(d)'} \neq \begin{pmatrix} 0 & |B_{12}|e^{i\theta} & 0 \\ |B_{21}|e^{-i\theta} & |B_{22}| & |B_{23}|e^{i\phi} \\ 0 & |B_{32}|e^{-i\phi} & |B_{33}| \end{pmatrix} \tag{5.1.11}$$

for any values of θ and ϕ. As is evident from our invariants, this statement does not depend on whether or not the initial $Y^{(u)}$ and $Y^{(d)}$ are (complex) symmetric.

For this model, we shall also illustrate explicitly the operations which enter into the proof of our theorem (2.17). In order to get the invertible matrix $\bar{T}$, we select and delete $N_{eq} - r_{_T} = 10 - 8 = 2$ rows from the T matrix for this model, given in eq. (5.1.3). This corresponds to deleting two elements of the vector w defined in (2.11) to get $\bar{w}$, i.e.

not rephasing the corresponding $Y_{jk}^{(f)}$. Given that we have constructed the full set of N_{ia} invariants with independent phases, we know the full set of allowed choices for elements to be rephased and correspondingly the choices for the $N_p = 2$ elements which will not be rephased. In this model, the rephasing equations for the elements comprising the complex invariants $P_{22,33}^{(d)}$ and $Q_{12,22}$ correspond to the rows (1,3,5,7) and (7,8,9,10) of the matrix T, respectively. Hence we must delete one row from the first set and one row from the second set; i.e. we have the following 15 choices of row pairs to delete: $(r1, r2)$, with $r1$ chosen from (1,3,5,7) and $r2$ chosen from (7,8,9,10), excluding, of course, the choice $r1 = r2 = 7$. For our specific illustration, we delete rows 5 and 7 and thus do not rephase $Y_{12}^{(d)}$ or $Y_{22}^{(d)}$. Note that if one were to try to remove two rows in a manner which would violate the phase constraints, one's error would immediately be signalled by a reduction in the rank of the resultant matrix. For example, if one were to try to remove rows 1 and 2, one would obtain a matrix with rank 7. Recall that this choice is forbidden since it would entail the implication that one could rephase all other elements of $Y^{(u)}$ and $Y^{(d)}$, which is false, since it would mean that one could in general rephase the argument of $Q_{12,22}$ to zero, contrary to fact. Note that, in principle, one may determine which rows may be deleted simply by testing the rank of the resultant matrix, without first constructing the complete set of complex invariants. However, obviously the more efficient procedure is to construct this set of invariants first, since once this is done, it is clear which sets of N_p rows can or cannot be deleted consistent with the phase constraints. Continuing with our explicit calculational example, having deleted the rows 5 and 7 from T, and thereby obtained an $r_{_T} \times 3N_G = 8$-row by 9-column matrix, we next delete $3N_G - r_{_T} = 1$ column from this matrix in order finally to get the square, $r_{_T} \times r_{_T} = 8 \times 8$ matrix $\bar{T}$. This corresponds to deleting one element of the vector v defined in eq. (2.10) to get $\bar{v}$. We find that any of the nine columns of T may be deleted in this

step; for definiteness, we pick the ninth. This yields the reduced matrix $\bar{T}$ for this model:

$$\bar{T}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.1.12}$$

(with determinant -1). By construction, this is invertible, and we find

$$\bar{T}_1^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{5.1.13}$$

As an illustration of our theorem on invariants, we discuss the 6'th order complex invariants for this model. Since we have already constructed as many independent complex invariants with independent arguments as there are unremovable phases, using quartic complex invariants, our theorem on invariants and its specific Corollary 1 imply that the argument of any 6'th, as well as higher order, complex invariant, are expressible in terms of the arguments of the quartic invariants. By explicit computation, we find that there is one complex 6'th order invariant (i.e. $N_{inv,6} = 1$):

$$Q^{(ddu)}_{32,23,12} = B_{32}B_{23}A_{12}B^*_{33}B^*_{12}A^*_{22} \tag{5.1.14}$$

Thus the total number of independent complex invariants for this model is $N_{inv} = N_{inv,4} + N_{inv,6} = 3$. The graphical representation of this complex invariant is shown in Fig. 3(a) (where here and elsewhere, lines which cross each other at a point where there is no $\odot$ or $\otimes$, as in the $Y^{(d)}_{22}$ position, do not mean that the invariant contains this element). Note that this 6'th order invariant satisfies the condition (i) of part (d) of the above theorem. It is

thus easy to find the explicit reduction; we obtain the factorization

$$\begin{aligned}
\arg(B_{32}B_{23}A_{12}B^*_{33}B^*_{12}A^*_{22}) &= \arg(B_{32}B_{23}A_{12}(B_{22}B^*_{22})B^*_{33}B^*_{12}A^*_{22}) \\
&= \arg(A_{12}B_{22}A^*_{22}B^*_{12}) + \arg(B_{32}B_{23}B^*_{22}B^*_{33}) \quad (5.1.15)
\end{aligned}$$

whence

$$\arg(Q^{(ddu)}_{32,23,12}) = \arg(Q_{12,22}) - \arg(P^{(d)}_{22,33}) \quad (5.1.16)$$

Indeed, in this case, we find the factorization

$$Q^{(ddu)}_{32,23,12} = |B_{22}|^{-2} Q_{12,22} P^{(d)}_{23,32} \quad (5.1.17)$$

Recall here that $P^{(d)}_{23,32} = P^{(d)*}_{22,33}$. Note also that according to our definition given above, $Q^{(ddu)}_{32,23,12}$ is still an irreducible 6'th order complex invariant because it is not expressible only as a product of lower-order complex invariants; there is also the $|B_{22}|^{-2}$ factor. Graphically, the insertion of the factor $B_{22}B^*_{22}$ in eq. (5.1.15) is equivalent to inserting a coincident arrow head and tail, or $\odot$ and $\otimes$ in the $Y^{(d)}_{22}$ position. Having done this, one sees immediately that the 6'th order invariant factorizes in the manner specified by eq. (5.1.17). In addition to using our theorem on invariants to show that the argument of $Q^{(ddu)}_{32,23,12}$ can be expressed in terms of those of quartic complex invariants, another way to see this is to use the Z matrix method discussed above. For this, we first define the vector ξ defined in eq. (3.31) for the present model:

$$(\xi)_1 = (\arg(P^{(d)}_{22,33}),\ \arg(Q_{12,22}),\ \arg(Q^{(ddu)}_{32,23,12}))^T \quad (5.1.18)$$

where the subscript on $(\xi)_1$ denotes the fact that this is the ξ vector for model 1. The ordering of the elements in the vector w was given in general after its definition, eq. (2.11), and explicitly for this model. Then Z is an $N_{inv} \times N_{eq} = 3$-row by 10-column matrix. From the definition (3.30) we calculate it to be

$$(Z)_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad (5.1.19)$$

where again the subscript on $(Z)_1$ refers to model 1. Since the elements A_{21}, A_{33}, and B_{21} do not occur in any complex invariants, they may be removed from the vector w to yield w_r, and correspondingly, one may remove the second, fourth, and sixth columns from Z_1 to form the reduced 4-row by 7-column matrix Z_r defined in (3.32),

$$(Z_r)_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{5.1.20}$$

Since we have only removed columns of zeroes, this does not change the rank of the matrix. We calculate that $rank((Z)_1) = rank((Z_r)_1) = 2$, in agreement with our theorem that $N_p = N_{ia}$ and the relation $N_{ia} = rank(Z)$ from eq. (3.34). We find further that removing the last row from $(Z)_1$ or $(Z_r)_1$ does not reduce the rank, which shows that the argument of $Q^{(ddu)}_{32,23,12}$ can be expressed in terms of those of $P^{(d)}_{22,33}$ and $Q_{12,22}$. The phase properties and complex invariants of the Yukawa matrices for this and the other realistic models to be discussed are summarized in Table 1. Having given a detailed discussion of the details of our analysis for this model, we next proceed with the others, which we will discuss more briefly.

5.2 Model 2

This model is defined by

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix} \tag{5.2.1}$$

and $Y^{(d)}$ as in model 1, i.e.

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \tag{5.2.2}$$

model	N_{eq}	r_T	$N_p = N_{ia}$	$N_{inv,4}$	P_4	Q_4	$N_{inv,6}$	P_6 or Q_6
1	10	8	2	2	$P^{(d)}_{22,33}$	$Q_{12,22}$	1	$Q^{(ddu)}_{32,23,12}$
2	11	8	3	4	$P^{(d)}_{22,33}$	$Q_{12,32},\ Q_{23,32},\ Q_{23,33}$	2	$Q^{(uud)}_{32,23,12},\ Q^{(ddu)}_{22,33,12}$
3	10	8	2	2	$P^{(d)}_{2233}$	$Q_{13,32}$	1	$Q^{(ddu)}_{22,33,13}$
4	10	8	2	2	$P^{(u)}_{22,33}$	$Q_{12,22}$	1	$Q^{(uud)}_{32,23,12}$
5	10	8	2	2	$P^{(u)}_{22,33}$	$Q_{13,22}$	1	$Q^{(uud)}_{32,13,22}$
6	12	8	4	7	$P^{(u)}_{22,33}$ $P^{(d)}_{22,33}$	$Q_{12,22},\ Q_{12,32},\ Q_{22,32},$ $Q_{22,33},\ Q_{23,33}$	4	$Q^{(uud)}_{22,33,12},\ Q^{(uud)}_{33,12,22}$ $Q^{(ddu)}_{22,33,12},\ Q^{(ddu)}_{33,12,22}$

Table 1: Summary of unremovable phases and complex invariants for the six realistic models of Yukawa matrices.

This model has $N_{z,u} = 4$, $N_{z,d} = 3$, and hence $N_{eq} = 11$. We calculate that T is the 11×9 matrix

$$T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.2.3}$$

with rank 8. Hence, from eq. (2.17), there are $N_p = N_{eq} - rank(T) = 3$ unremovable phases in the $Y^{(f)}$, $f = u, d$. By part (c) of our theorem on invariants, there are correspondingly $N_{ia} = N_p = 3$ independent phases of complex invariants. We actually find that there are $N_{inv,4} = 4$ independent complex quartic invariants in this model. (Recall from eq. (3.28) that in general there may be more independent complex invariants than independent arguments of invariants.) The four independent complex quartic invariants are $P^{(d)}_{22,33}$ as in model 1, together with

$$Q_{12,32} = A_{12} B_{32} A^*_{32} B^*_{12} \tag{5.2.4}$$

$$Q_{23,32} = A_{23} B_{32} A^*_{33} B^*_{22} \tag{5.2.5}$$

and

$$Q_{23,33} = A_{23} B_{33} A^*_{33} B^*_{23} \tag{5.2.6}$$

Of these, the arguments of the first, third, and fourth invariants are related according to

$$\arg(P^{(d)}_{22,33}) + \arg(Q_{23,32}) - \arg(Q_{23,33}) = 0 \tag{5.2.7}$$

so that, as required by our theorem, the number of independent arguments of invariants is $N_{ia} = 3$. One can also determine the number N_{ia} without using part (c) of our theorem and Corollary 1 on invariants by calculating the matrix Z defined in eq. (3.30) and computing its rank. In the present case, it is sufficient, and simpler, to deal with the matrix Z_4 defined

in eq. (3.36) for quartic complex invariants. As defined in eq. (3.35) for $2n = 4$, the vector ξ_4 of arguments of quartic invariants has dimension $N_{inv,4} = 4$ and is

$$(\xi_4)_1 = (\arg(P^{(d)}_{22,33}),\ \arg(Q_{12,32}),\ \arg(Q_{23,32}),\ \arg(Q_{23,33}))^T \tag{5.2.8}$$

where the subscript 2 on $(\xi_4)_2$ refers to model 2. Using (3.36) and (2.11) with the specified ordering of elements in w_4 (and $\eta^{(f)}_{jk} = 0$ as noted before), we then calculate the $N_{inv,4} \times N_{eq}$ = 4-row by 11-column Z_4 matrix as

$$(Z_4)_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{5.2.9}$$

This matrix has rank 3, so that the three independent arguments among the four quartic complex invariants account for the full set of three unremovable phases in this model. By selecting and deleting rows of $(Z_4)_2$, one easily establishes which of these quartic invariants have linearly dependent arguments; deleting row 1, 3, or 4 does not reduce the rank, while deleting row 2 yields a matrix with rank 2. This means that any one of the sets $(Q_{12,32},\ Q_{23,32},\ Q_{23,33})$, $(P^{(d)}_{22,33},\ Q_{12,32},\ Q_{23,33})$, and $(P^{(d)}_{22,33},\ Q_{12,32},\ Q_{23,32})$ constitutes a complete set of $N_{ia} = 3$ complex invariants with linearly independent arguments, while the arguments of the invariants in the set $(P^{(d)}_{22,33},\ Q_{23,32},\ Q_{23,33})$ are linearly dependent, in agreement with the explicit calculation (5.2.7). We will take the three independent unremovable phases in this model to be $\arg(P^{(d)}_{22,33})$, $\arg(Q_{12,32})$, and $\arg(Q_{23,32})$. Since the elements A_{21} and B_{21} do not occur in any complex invariants, they may be rephased freely, and one may also form the vector w_r and corresponding matrix $Z_{r,4}$ which is just Z_4 with the second and seventh columns of zeroes removed. The constraints on rephasing from $P^{(d)}_{22,33}$ are (i) and (ii) as in model 1, and (iii) none of the sets

$$S_{Q_{12,32}} = \{A_{12}, A_{32}, B_{12}, B_{32}\} \tag{5.2.10}$$

$$S_{Q_{23,32}} = \{A_{23}, A_{33}, B_{22}, B_{32}\} \tag{5.2.11}$$

and

$$S_{Q_{23,33}} = \{A_{23}, A_{33}, B_{23}, B_{33}\} \tag{5.2.12}$$

can be made simultaneously real. In particular, if $Y^{(u)}$ is made real, then none of the sets $\{B_{12}, B_{32}\}$, $\{B_{32}, B_{22}\}$, and $\{B_{23}, B_{33}\}$ can be made simultaneously real. As in model 1, because of the constraint from $P^{(d)}_{22,33}$, $Y^{(d)}$ cannot in general be made real or hermitian. An example of a rephasing of the Yukawa matrices allowed by these constraints is $Y^{(u)}$ real,

$$Y^{(u)'} = \begin{pmatrix} 0 & |A_{12}| & 0 \\ |A_{21}| & 0 & |A_{23}| \\ 0 & |A_{32}| & |A_{33}| \end{pmatrix} \tag{5.2.13}$$

and

$$Y^{(d)'} = \begin{pmatrix} 0 & |B_{12}|e^{-i\,\arg(Q_{12,32})} & 0 \\ |B_{21}| & |B_{22}|e^{-i\,\arg(Q_{23,32})} & |B_{23}| \\ 0 & |B_{32}| & |B_{33}|e^{i(\arg(P^{(d)}_{22,33})+\arg(Q_{23,32}))} \end{pmatrix} \tag{5.2.14}$$

The construction of $\bar{T}$ from T is performed in a manner similar to that discussed for model 1.

As in model 1, since we have already constructed as many independent complex invariants with independent arguments as there are unremovable phases, $N_{ia} = N_p = 3$, using quartic complex invariants, our theorem on invariants and its Corollary 1 imply that the arguments of any 6'th, as well as higher order, complex invariants, are expressible in terms of the arguments of the quartic invariants and hence yield no new phase constraints. We find that in the present model there are two different 6'th order complex invariants (so $N_{inv,6} = 2$):

$$Q^{(uud)}_{32,23,12} = A_{32}A_{23}B_{12}A^*_{33}A^*_{12}B^*_{22} \tag{5.2.15}$$

and

$$Q^{(ddu)}_{22,33,12} = B_{22}B_{33}A_{12}B^*_{23}B^*_{12}A^*_{32} \tag{5.2.16}$$

The total number of independent complex invariants in this model is therefore $N_{inv} = N_{inv,4} + N_{inv,6} = 6$. The explicit expressions for the arguments of these 6'th order complex invariants, in terms of quartic complex invariants, are

$$\arg(Q^{(uud)}_{32,23,12}) = \arg(Q_{23,32}) - \arg(Q_{12,32}) \tag{5.2.17}$$

and

$$\arg(Q^{(ddu)}_{22,33,12}) = \arg(P^{(d)}_{22,33}) + \arg(Q_{12,32}) \tag{5.2.18}$$

Again, a different way of showing that the 6'th order complex invariants yield no new constraints is by analyzing the full Z matrix for the model. The method should be clear from our previous example, so we omit the details.

Parenthetically, we consider the special case of this model when $Y^{(d)}_{22} = B_{22} = 0$. Here $Y^{(u)}$ and $Y^{(d)}$ have the same form. Although current interest is in models of Yukawa matrices which apply at the SUSY GUT level, it should be noted that the further specialization in which one takes $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$ is formally the same as one of the early works on quark mass matrices [10]. Of course in that work the form was hypothesized to apply near the electroweak level in a (non-supersymmetric) $SU(2)_L \times SU(2)_R \times U(1)$ electroweak gauge model, not a SUSY GUT. As is well known, whether one considers the symmetric form with $B_{22} = 0$ to apply at the electroweak level or at the SUSY GUT level with MSSM evolution equations, it is currently disfavored by experiment, because of the large value which it predicts for V_{cb}, given the lower bound on top quark mass indicated by both direct limits from CDF and D0 [11] and by precision fits to LEP data [12]. It is, however, of historical interest. In this case, $N_{eq} = 10$, $rank(T) = 8$, whence $N_p = 2$. Of the four quartic invariants for model 2, two vanish for this special case: $P^{(d)}_{22,33} = 0$ and $Q_{23,32} = 0$. The other quartic complex invariants, $Q_{13,32}$ and $Q_{23,33}$, remain, so that $N_{inv,4} = 2$. The phases of these are independent, so these two quartic invariants account for both of the two unremovable phases and resultant phase constraints. The single 6'th order complex invariant for model 2 vanishes for this special case, which therefore has no nonzero complex 6'th order invariants.

5.3 Model 3

The third model is given by

$$Y^{(u)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & 0 \\ A_{31} & 0 & A_{33} \end{pmatrix} \tag{5.3.1}$$

with $Y^{(d)}$ as in model 1:

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \tag{5.3.2}$$

Special cases of this model for which $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$ were studied in Refs. [14] and [7]; in Ref. [14], the further assumptions $|A_{22}| = |A_{13}|$ and $|B_{23}| = 2|B_{22}|$ were made. The calculation of T should be clear from our previous examples, so we just mention that it is a 10×9 matrix and we find its rank to be 8. Hence, there are $N_p = 2$ unremovable phases in the $Y^{(f)}$, $f = u, d$. We find $N_{inv,4}$ independent complex quartic invariants; these are $P^{(d)}_{22,33}$ as in the previous models, and

$$Q_{13,32} = A_{13} B_{32} A^*_{33} B^*_{12} \tag{5.3.3}$$

These two complex quartic invariants clearly have independent phases, so that they fully account for the two unremovable phases in the model: $N_{inv,4} = N_{ia} = 2$. The phase constraint from $P^{(d)}_{22,33}$ is the same as in models 1 and 2 [11] The phase constraint from $Q_{13,32}$ is that the set of elements

$$S_{Q_{13,32}} = \{A_{13}, A_{31}, B_{12}, B_{32}\} \tag{5.3.4}$$

cannot be made simultaneously real. The elements A_{31}, A_{33}, and B_{21} do not occur in any complex invariants and may be rephased freely. An example of an allowed form for the Yukawa matrices after rephasing of fermion fields is $Y^{(u)}$ real,

$$Y^{(u)\prime} = \begin{pmatrix} 0 & 0 & |A_{13}| \\ 0 & |A_{22}| & 0 \\ |A_{31}| & 0 & |A_{33}| \end{pmatrix} \tag{5.3.5}$$

and

$$Y^{(d)\prime} = \begin{pmatrix} 0 & |B_{12}| e^{-i\,\arg(Q_{13,32})} & 0 \\ |B_{21}| & |B_{22}| e^{i\,\arg(P^{(d)}_{22,33})} & |B_{23}| \\ 0 & |B_{32}| & |B_{33}| \end{pmatrix} \tag{5.3.6}$$

By our theorem on invariants and its specific Corollary 1, any 6'th order complex invariants have arguments which can be expressed in terms of those of the quartic complex invariants. We find that the model has one 6'th order complex invariant (so $N_{inv,6} = 1$):

$$Q^{(ddu)}_{22,33,13} = B_{22} B_{33} A_{13} B^*_{23} B^*_{12} A^*_{33} \tag{5.3.7}$$

[11] Ref. [14] actually presents a form in which the submatrix (5.1.7) of $Y^{(d)}$ is real. Such a form is not, in general, possible to achieve by any fermion rephasings, as we have already discussed above in the context of model 1.

whose argument is expressed in terms of those of the quartic invariants as

$$\arg(Q^{(ddu)}_{22,33,13}) = \arg(P^{(d)}_{22,33}) + \arg(Q_{13,32}) \tag{5.3.8}$$

The total number of complex invariants is $N_{inv} = N_{inv,4} + N_{inv,6} = 3$.

5.4 Model 4

The fourth model is defined by

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix} \tag{5.4.1}$$

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix} \tag{5.4.2}$$

This model can be obtained from model 1 by the interchange of $Y^{(u)}$ and $Y^{(d)}$. Using this fact, one can immediately determine all of the phase properties and invariants of the model from those of model 1: $N_p = N_{ia} = 2$, and the two independent quartic complex invariants are

$$P^{(u)}_{22,33} = A_{22}A_{33}A^*_{32}A^*_{32} \tag{5.4.3}$$

and $Q_{12,22}$ as given in (4.18). These fully account for all phase constraints in the model, which are given by those for model 1 with the interchange of $Y^{(u)} \leftrightarrow Y^{(d)}$ ($A_{jk} \leftrightarrow B_{jk}$). An allowed form for the Yukawa matrices after rephasing is

$$Y^{(u)'} = \begin{pmatrix} 0 & |A_{12}|e^{-i\,\arg(Q_{12,22})} & 0 \\ |A_{21}| & |A_{22}|e^{i\,\arg(P^{(u)}_{22,33})} & |A_{23}| \\ 0 & |A_{32}| & |A_{33}| \end{pmatrix} \tag{5.4.4}$$

and

$$Y^{(d)'} = \begin{pmatrix} 0 & |B_{12}| & 0 \\ |B_{21}| & |B_{22}| & 0 \\ 0 & 0 & |B_{33}| \end{pmatrix} \tag{5.4.5}$$

The model has one 6'th order complex invariant, $Q^{(uud)}_{32,23,12}$ (given explicitly in (5.2.15) for model 2). The argument of this 6'th order invariant is expressed in terms of those of the two

complex quartic invariants by the relation

$$\arg(Q^{(uud)}_{32,23,12}) = -\arg(Q_{12,22}) - \arg(P^{(u)}_{22,33}) \tag{5.4.6}$$

As this illustrates, it is obvious that if the same 6'th order complex invariant occurs in two different model and if in both cases its argument is expressible in terms of those of the respective complex quartic invariants for these models, then these expressions will be different, if, as will generally be the case, the complex quartic invariants are different for the two models. Specifically, one sees that the relation (5.4.6) in the present model differs from (5.1.16) in model 2.

The special case of model 4 in which $|Y^{(f)}_{jk}| = |Y^{(f)}_{kj}|$, $f = u, d$ and $Y^{(u)}_{22} = A_{22} = 0$ [13] is not included in ref. [7] among its experimentally acceptable forms because of the large values of $|V_{cb}|$ which it yields and, if other quantities are used as inputs to get a prediction for the top quark mass m_t, the high values of m_t which it predicts. However, this special case, and also its generalization with $|Y^{(f)}_{jk}|$ not necessarily equal to $|Y^{(f)}_{kj}|$ has several interesting properties concerning phases and rephasing invariants. First, since N_{eq} is reduced to 9 while $rank(T)$ remains at 8, there is only one unremovable phase, $N_p = 1$. As is evident from eqs. (5.4.3) and (4.18), both of the quartic complex invariants $P^{(u)}_{22,33}$ and $Q_{12,22}$ vanish, so that there are no quartic complex invariants; $N_{inv,4} = 0$. Therefore, in contrast to the general model 4 with nonzero A_{22}, in this special case, (i) the only complex invariant is the 6'th order invariant $Q^{(uud)}_{32,23,12}$ given above in (5.2.15); (ii) as is evident from (5.2.17), the argument of this 6'th order invariant cannot be expressed in terms of any lower complex invariants, since there are none. Thus this special case provides an example of a model in which the lowest-order complex invariant(s) are of 6'th rather than 4'rth order, and $N_{inv} = N_{inv,6} = 1$. The phase constraint yielded by this invariant is that the set of elements $\{A_{12}, A_{23}, A_{32}, A_{33}, B_{12}, B_{22}\}$ cannot be made simultaneously This allows $Y^{(u)}$ to be made real; if it is, then $\{B_{12}, B_{22}\}$ cannot be made real. The constraint also allows $Y^{(d)}$ to be made real, and if it is, then $\{A_{12}, A_{23}, A_{32}, A_{33}\}$ cannot all be made real.

5.5 Model 5

This model is given by

$$Y^{(u)} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{5.5.1}$$

and $Y^{(d)}$ as in (5.4.2),

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix} \tag{5.5.2}$$

Here $N_{eq} = 10$, and the T matrix is 10×9 with rank 8, so that $N_p = 2$. We find two independent complex quartic invariants: $P^{(u)}_{22,33}$ as in (5.4.3), and

$$Q_{13,22} = A_{13} B_{22} A^*_{23} B^*_{12} \tag{5.5.3}$$

Since the two complex quartic invariants have independent phases, they fully account for all of the phase constraints. These constraints are that it is not possible to render the respective elements comprising the sets

$$S_{P^{(u)}_{22,33}} = \{A_{22}, A_{23}, A_{32}, A_{33}\} \tag{5.5.4}$$

or

$$S_{Q_{13,22}} = \{A_{13}, A_{23}, B_{12}, B_{22}\} \tag{5.5.5}$$

simultaneously real by any fermion rephasings. These constraints allow one to render $Y^{(d)}$ real. If one does this, then one of the two unremovable phases must reside in the submatrix

$$\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} \tag{5.5.6}$$

formed by the elements of set (5.5.4) and the other in the set $\{A_{13}, A_{23}\}$. The elements A_{31}, B_{21}, and B_{33} do not occur in any complex invariants and hence may be rephased freely. A possible form for the Yukawa matrices after rephasing is

$$Y^{(u)\prime} = \begin{pmatrix} 0 & 0 & |A_{13}|e^{i\ \arg(Q_{13,22})} \\ 0 & |A_{22}|e^{i\ \arg(P^{(u)}_{22,33})} & |A_{23}| \\ |A_{31}| & |A_{32}| & |A_{33}| \end{pmatrix} \tag{5.5.7}$$

$$Y^{(d)\prime} = \begin{pmatrix} 0 & |B_{12}| & 0 \\ |B_{21}| & |B_{22}| & 0 \\ 0 & 0 & |B_{33}| \end{pmatrix} \tag{5.5.8}$$

Since the quartic complex invariants already contain all of the unremovable phases, it follows from our theorem on invariants and its Corollary 1 that any 6'th order complex invariant yields no new phase constraint. We find that there is one 6'th order complex invariant,

$$Q^{(uud)}_{32,13,22} = A_{32}A_{13}B_{22}A^*_{33}A^*_{22}B^*_{12} \tag{5.5.9}$$

Its argument is expressed in terms of those of quartic complex invariants as

$$\arg(Q^{(uud)}_{32,13,22}) = -\arg(P^{(u)}_{22,33}) + \arg(Q_{12,22}) \tag{5.5.10}$$

The model thus has $N_{inv,4} = 2$, $N_{inv,6} = 1$, and $N_{inv} = 3$.

5.6 Model 6

We have noted above that, to the extent that one understands the phenomenological implications of string theories, they do not appear to yield symmetric Yukawa matrices. This is, indeed, part of the reason that we have listed the previous five forms for these matrices in a form which is not, in general, symmetric. Here, as a further illustration of our methods, we determine the unremovable phases and associated complex invariants for a generalization of one specific model of Yukawa matrices inspired by a particular 4D superstring construction and sketched in [8] [12] The model is defined by

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix} \tag{5.6.1}$$

$$Y^{(d)} = \begin{pmatrix} 0 & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \tag{5.6.2}$$

As can be seen, if one takes $A_{22} = 0$, this reduces to model 4. The present model has $N_{eq} = 12$. We calculate the 12×9 T matrix as in the previous cases and find that it has

[12] Ref. [8] does not give any discussion of the number of unremovable phases or which elements of the Yukawa matrices can be made real.

$rank(T) = 8$, so that $N_p = 4$. We find that there are $N_{inv,4} = 7$ quartic complex invariants, two of P_4 type and five of Q_4 type. These are $P^{(u)}_{22,33}$, $P^{(d)}_{22,33}$, $Q_{12,22}$, $Q_{12,32}$, $Q_{22,32}$, $Q_{22,33}$, $Q_{22,33}$, and $Q_{23,33}$. The ordering of the arguments of the complex quartic invariants in the vector ξ_4 may be taken to be the same as in the list just given. The corresponding Z_4 matrix for these order 4 invariants is the $N_{inv,4} \times N_{eq} = 7$-row by 12-column matrix

$$(Z_4)_6 = \begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{5.6.3}$$

(where the second subscript refers to model 6). We find that $rank((Z_4)_6) = 4$, so that the quartic complex invariants fully account for all of the $N_p = N_{ia} = 4$ unremovable phases. Removing the last three rows of $(Z_4)_6$ does not reduce the rank, which shows that the arguments of the first four quartic complex invariants in $(\xi_4)_6$ are linearly independent, so that we may take these complex invariants, $P^{(u)}_{22,33}$, $P^{(d)}_{22,33}$, $Q_{12,22}$, and $Q_{12,32}$, as the set with independent arguments. Each of these complex invariants implies a phase constraint that the corresponding set of elements cannot be made simultaneously real. A possible form of the Yukawa matrices, after rephasing, which is consistent with the phase constraints, is

$$Y^{(u)\prime} = \begin{pmatrix} 0 & |A_{12}|e^{i\ \arg(Q_{12,22})} & 0 \\ |A_{21}| & |A_{22}| & |A_{23}|e^{i(-\arg(P^{(u)}_{22,33})-\arg(Q_{12,22})+\arg(Q_{12,32}))} \\ 0 & |A_{32}|e^{i(\arg(Q_{12,22})-\arg(Q_{12,32}))} & |A_{33}| \end{pmatrix} \tag{5.6.4}$$

$$Y^{(d)\prime} = \begin{pmatrix} 0 & |B_{12}| & 0 \\ |B_{21}| & |B_{22}| & |B_{23}|e^{-i\ \arg(P^{(d)}_{22,33})} \\ 0 & |B_{32}| & |B_{33}| \end{pmatrix} \tag{5.6.5}$$

The model has $N_{inv,6} = 4$ independent 6'th order complex invariants:

$$Q^{(uud)}_{22,33,12} = A_{22}A_{33}B_{12}A^*_{23}A^*_{12}B^*_{32} \tag{5.6.6}$$

$$Q^{(uud)}_{33,12,22} = A_{33}A_{12}B_{22}A^*_{32}A^*_{23}B^*_{12} \tag{5.6.7}$$

$Q^{(ddu)}_{22,33,12}$ as given explicitly before in (5.2.16) for model 2, and

$$Q^{(ddu)}_{33,12,22} = B_{33}B_{12}A_{22}B^*_{32}B^*_{23}A^*_{12} \tag{5.6.8}$$

Thus, $N_{inv} = N_{inv,4}+N_{inv,6} = 11$ for this model. By our theorem on invariants and its Corollary 1, the arguments of all of these 6'th order complex invariants are expressible in terms of those of the four quartic complex invariants with independent arguments. The methods have been fully illustrated in our previous models, so we omit the explicit expressions.

5.7 Model 7

We proceed to mention two toy models briefly to illustrate certain theoretical points. We have exhibited a special case of model 4 for which the only complex invariant occurs at 6'th order and is of Q_6 type. It is of interest to exhibit a form for the Yukawa matrices which again has only a 6'th order complex invariant, but for which this is of P_6 rather than Q_6 type. We have constructed such a model:

$$Y^{(u)} = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & 0 & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \tag{5.7.1}$$

and $Y^{(d)}$ as in (5.4.2),

$$Y^{(d)} = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & 0 & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} \tag{5.7.2}$$

This model thus has $N_{eq} = 9$. We calculate the T matrix and find that it has $rank(T) = 8$, so that $N_p = 1$. By design, this model has no quartic complex invariants, and a 6'th order complex invariant

$$P^{(d)}_{11,23,32} = B_{11}B_{23}B_{32}B^*_{12}B^*_{21}B^*_{33} \tag{5.7.3}$$

The graphical representation of this invariant is given in Fig. 3(b) for $f = d$. The elements of $Y^{(u)}$ are unconstrained by any phase invariants and, in particular, $Y^{(u)}$ can be made real. The phase constraint from $P^{(d)}_{11,23,32}$ forbids $Y^{(d)}$ from being made real or hermitian. A possible form for $Y^{(d)}$ after rephasing is

$$Y^{(d)'} = \begin{pmatrix} |B_{11}| & |B_{12}| & 0 \\ |B_{21}| & 0 & |B_{23}|e^{i\ \arg(P^{(d)}_{11,23,32})} \\ 0 & |B_{32}| & |B_{33}| \end{pmatrix} \tag{5.7.4}$$

5.8 Model 8

We noted in section 2 that the rank of the T matrix is 8 for all of the realistic models that we have studied, but that, in principle, $rank(T)$ may be less than 8. We present here a toy model to illustrate this. The model also provides an illustration of the inequality (4.23), that the number of unremovable phases in the Yukawa matrices is $\geq$ the number of phases in the CKM quark mixing matrix, V. Although formally the model has $N_G = 3$, it is of a degenerate type in which the third generation is decoupled from the first two in both the up and down quark sectors. Hence, in some ways it acts more like an $N_G = 2$ model. The Yukawa matrices are

$$Y^{(u)} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix} \tag{5.8.1}$$

$$Y^{(d)} = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{33} \end{pmatrix} \tag{5.8.2}$$

So that $N_{eq} = 10$. The T matrix is

$$T_8 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.8.3}$$

with $rank(T_8) = 7$, so that $N_p = N_{eq} - rank(T) = 3$. The model has $N_{inv} = N_{inv,4} = 6$ complex invariants, which are precisely those listed in eqs. (4.14) for $f = u$ and d and in eqs. (4.15)-(4.18) in section 4 for the $N_G = 2$ model with all nonzero arbitrary elements. The $N_{ia} = N_p = 3$ complex invariants among these with independent arguments may be taken to be $P^{(u)}_{11,22}$, $P^{(d)}_{11,22}$, and $Q_{11,22}$. Note that this model also is like an $N_G = 2$ model in that there is no CP violation. If one chooses to consider only the mutually coupled part of the model, then the Yukawa matrices would reduce to the upper left 2×2 submatrices of

(5.8.1) and (5.8.2), the model would reduce to a quasi $N_G = 2$ model, and T would become an 8×9 matrix with rank 5. Hence for the mutually coupled sector of the model, the rank of T would obey the relation that $rank(T) = rank(T)_{max} = 3N_G - 1$ (whence $rank(T) = 5$ for $N_G = 2$) analogous to the rank 8 nature of the true $N_G = 3$ models. When regarded as a degenerate $N_G = 3$ case, this model also constitutes an exception to the empirical rule that $N_p = (N_p)_{max} - N_z$, since $(N_p)_{max} = 10$ for $N_G = 3$ and $N_z = 8$. However, again, if one considered only the mutually coupled sector of the model, then it would be a quasi $N_G = 2$ model, so that $(N_p)_{max} = 3$, $N_z = 0$, and the empirical relation would be obeyed. Such subtle distinctions are not necessary for the realistic models studied here.

6 Conclusions

The goal of understanding quark masses and quark mixing remains one of the most important outstanding problems in particle physics. When one constructs a model of the quark Yukawa matrices, it is necessary to determine how many unremovable phases there are in $Y^{(u))}$ and $Y^{(d)}$, and to determine where these phases may be assigned, and which elements may be rendered real by rephasings of fermion fields. We previously reported a general solution to this problem [2] involving several theorems and a related analysis of complex rephasing-invariant products of elements of the Yukawa matrices. In the present paper we have given a detailed discussion of our methods and results. We have also presented new results on phases and complex invariants for the case of arbitrary N_G, the number of fermion generations, since this elucidates their theoretical properties and gives further insight into the case of physical interest, $N_G = 3$. Finally, we have given a detailed application of our methods to currently viable models.

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References

[1] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49** (1973) 652.

[2] A. Kusenko and R. Shrock, Phys. Rev. Lett., recommended for publication (hep-ph/9310307).

[3] A. Kusenko and R. Shrock, Phys. Lett. B, in press (hep-ph/9311307). Further details and applications are given in A. Kusenko and R. Shrock, ITP-SB-93-68, to appear soon.

[4] I-H. Lee, J. Shigemitsu, and R. Shrock, Nucl. Phys. **B330** (1990) 225; **B334** (1990) 265; R. Shrock, *Lattice Yukawa Models*, in *Quantum Fields on the Lattice*, ed. M. Creutz (World Scientific, Singapore, 1992).

[5] For recent reviews, see H. Haber, SCIPP-92-33 (hep-ph/9306207); SCIPP-93-22 (hep-ph/9308209); G. Kane et al., UM-TH-93-24 (hep-ph-9312272) and references therein.

[6] For some recent reviews discussing these aspects of strings, see, e.g., B. Greene, in *Proceedings of the XVI International Symposium on Lepton-Photon Interactions*, to be published; M. Dine, in *Proceedings of the XXVI International Conference on High Energy Physics*, (A.I.P., New York, 1993), vol. 2, p. 1386, and references therein.

[7] P. Ramond, R. Roberts, and G. G. Ross, Nucl. Phys. **B406** (1993) 19 (hep-ph/9303320).

[8] A. Faraggi and E. Halyo WIS-93-35-PH (hep-ph/9306235).

[9] A. Kusenko and R. Shrock, Phys. Rev. D, in press (hep-ph/9307344).

[10] H. Fritzsch, Phys. Lett. B**73** (1978) 317; Nucl. Phys. B**155** (1979) 182.

[11] P. Tipton (CDF Collaboration) and N. Hadley (D0 Collaboration) in the *Proceedings of the XVI International Symposium on Lepton-Photon Interactions, 1993*, Cornell, to be published.

[12] R. Rolandi, in *Proceedings of the XXVI International Conference on High Energy Physics, 1992*, Dallas, (A.I.P., New York, 1993), v. 1, p. 56; M. Swartz, in *Proceedings of the XVI International Symposium on Lepton-Photon Interactions, 1993*, Cornell, to be published; D. Buskulic et al., Z. Phys. **C60** (1993) 71.

[13] S. Dimopoulos, L. Hall, and S. Raby, Phys. Rev. D**45** (1992) 4195.

[14] G. Giudice, Mod. Phys. Lett. A **7** (1992) 2429.

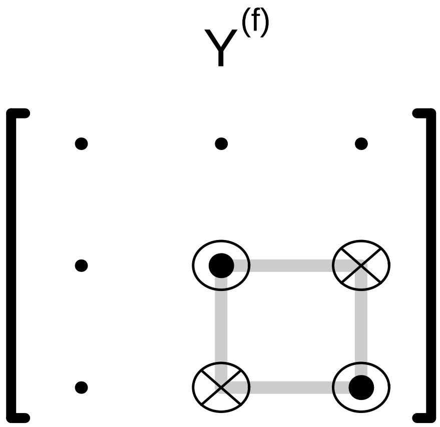

Fig. 1(a), $P^{(f)}_{22,33}$

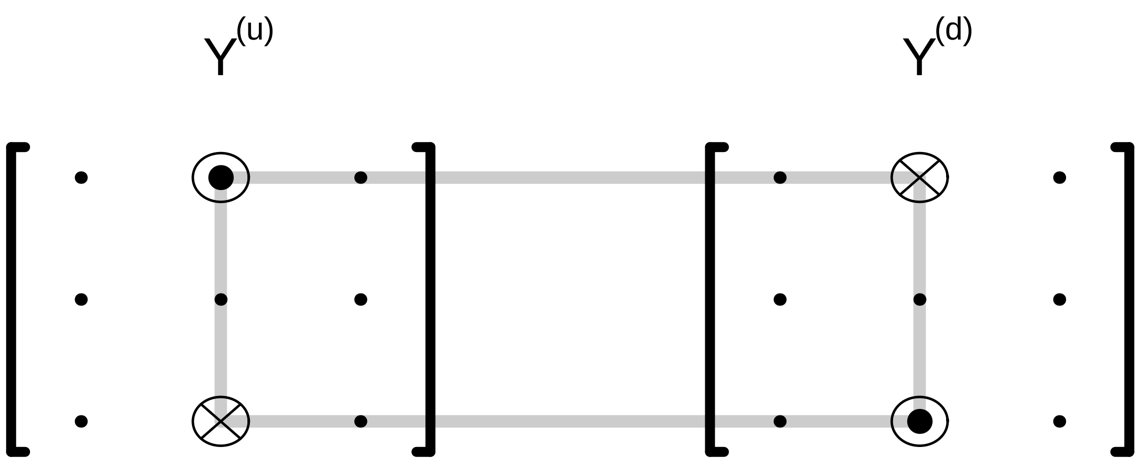

Fig. 1(b), $Q_{12,22}$

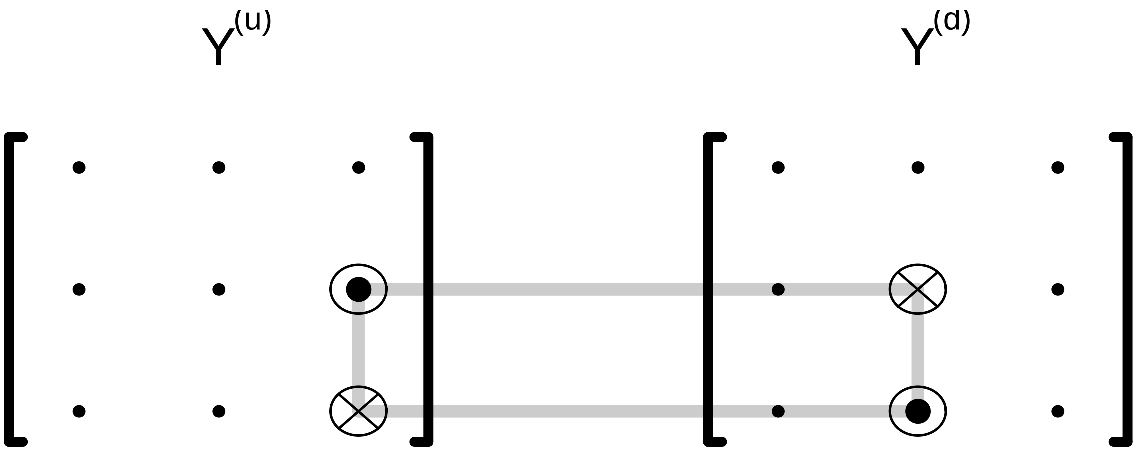

Fig. 1(c), $Q_{23,32}$

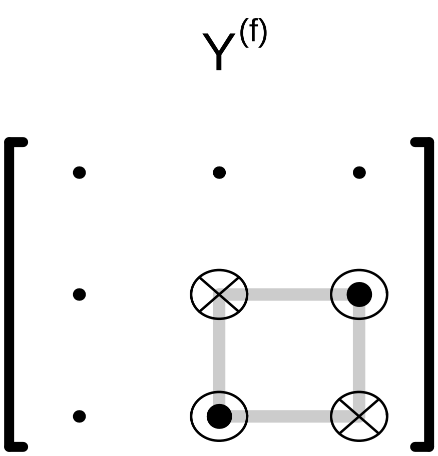

Fig. 2(a), $P^{(f)*}_{22,33}=P^{(f)}_{23,32}$

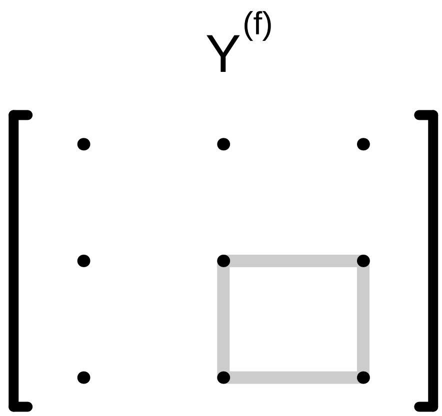

Fig. 2(b), $\{P^{(f)}_{22,33}, P^{(f)*}_{22,33}\}$

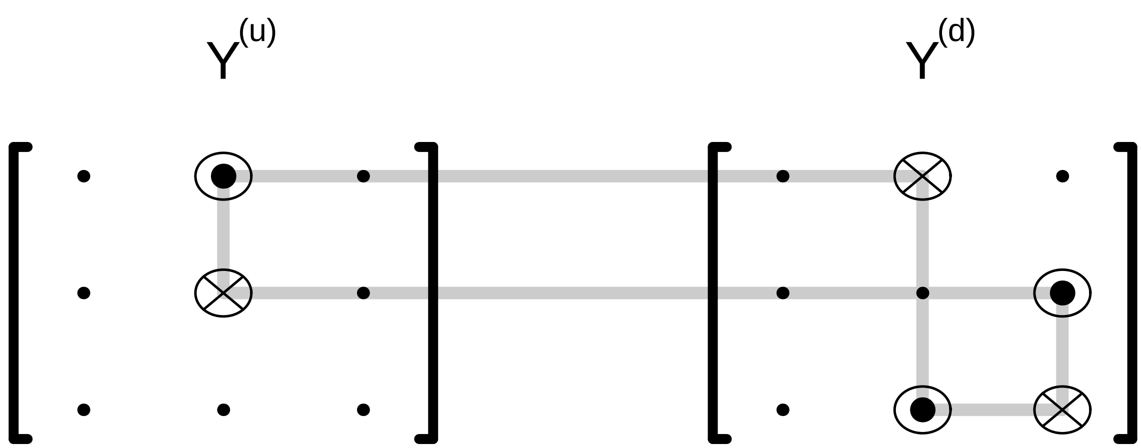

Fig. 3(a), $Q^{(ddu)}_{32,23,12}$

$Y^{(f)}$

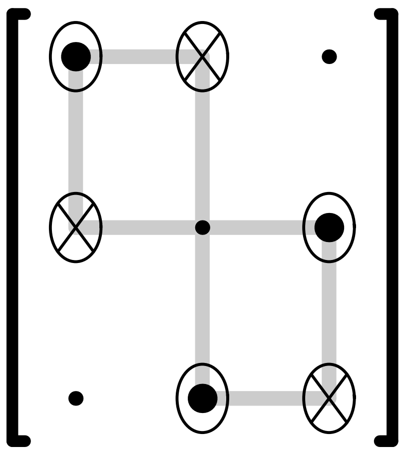

Fig. 3(b), $P^{(f)}_{11,23,32}$